

# ON PARTIALLY WRAPPED FUKAYA CATEGORIES

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**ABSTRACT.** We define a new class of symplectic objects called “stops”, which roughly speaking are Liouville hypersurfaces in the boundary of a Liouville domain. Locally, these can be viewed as pages of a compatible open book. To a Liouville domain with a collection of disjoint stops, we assign an  $A_\infty$ -category called its partially wrapped Fukaya category. An exact Landau-Ginzburg model gives rise to a stop, and the corresponding partially wrapped Fukaya category is meant to agree with the Fukaya category one is supposed to assign to the Landau-Ginzburg model. As evidence, we prove a formula that relates these partially wrapped Fukaya categories to the wrapped Fukaya category of the underlying Liouville domain. This operation is mirror to removing a divisor.

In the process, we construct continuation functors without cascades, which should be of independent interest.

## 1. INTRODUCTION

**1.1. Stops.** Our basic goal is to use geometric data to enhance the wrapped Fukaya category of a Liouville to the so-called “partially wrapped Fukaya category”. Recall that the wrapped Fukaya category [9] of a Liouville domain  $(M, \lambda)$  is the  $A_\infty$  category  $\mathcal{W}(M)$  whose objects are properly embedded exact Lagrangians, and whose morphism spaces are Floer cochain complexes generated, roughly speaking, by interior intersections and positive-time Reeb chords on the boundary. For this paper we take all Floer complexes to have coefficients in a field  $\mathbb{K}$  of characteristic 2, though in general  $\mathcal{W}(M)$  can be defined over  $\mathbb{Z}$ . The wrapped Fukaya category was introduced to satisfy mirror symmetry for open Calabi-Yau manifolds, where it has proven extremely successful and notoriously difficult to compute.

To enhance it, we introduce the notion of a *stop*. Roughly speaking, a stop  $\sigma$  in a Liouville domain  $(M, \lambda)$  is a hypersurface with boundary of the boundary  $\partial M$  such  $(\sigma, \lambda|_\sigma)$  is itself a Liouville domain. For an appropriate presentation of  $\mathcal{W}(M)$ , intersection number of a Reeb chord with  $\sigma$  gives a filtration on the wrapped Floer cochain complexes, and we refer to the zero-filtered part as the *partially wrapped* Floer cochain complex. The *partially wrapped Fukaya category* of  $(M, \lambda, \sigma)$ , denoted  $\mathcal{W}_\sigma(M)$ , is then the unital, non-full  $A_\infty$  subcategory of  $\mathcal{W}(M)$  consisting of those objects which avoid  $\sigma$  and those morphisms which have intersection number zero with  $\sigma$ .

The main result of this paper, Theorem 4.10, characterizes the inclusion functor  $\mathcal{W}_\sigma(M) \rightarrow \mathcal{W}(M)$ . Here, the key technical assumption is that a stop, treated as a Liouville domain, is *strongly nondegenerate*. For a Liouville domain  $F$ , this means that the following three conditions hold.

- (1) The symplectic cohomology  $SH^*(F)$  is not zero.
- (2)  $F$  is nondegenerate in the sense of Ganatra [19], which means that it admits a finite collection of Lagrangians satisfying Abouzaid’s generation criterion [2]. Ganatra showed that this is equivalent to the condition that the open-closed map

$$\mathcal{OC}: HH_*(\mathcal{W}(F)) \rightarrow SH^{*+n}(F)$$

from the Hochschild homology of the wrapped Fukaya category of  $F$  to the symplectic cohomology of  $F$  is an isomorphism.

- (3) The action of  $\mathcal{OC}^{-1}(\mathbf{1})$  is zero, where  $\mathbf{1}$  is the multiplicative unit of symplectic cohomology. Here, the action map on Hochschild homology is induced from the action filtration on Floer cochain groups; for a precise definition, see Section 4.1.

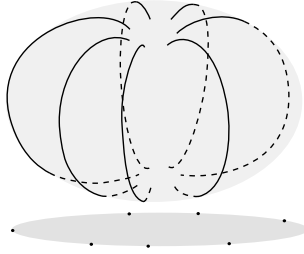


FIGURE 1. The prototypical pumpkin domain.  $\partial M$  is equipped with a finite collection of pages from an open book.

In particular, any Riemann surface or cotangent bundle is strongly nondegenerate. The generation argument in the upcoming paper [20] will show, more generally, that any Liouville domain admitting a Lagrangian skeleton with arboreal singularities [30] is strongly nondegenerate.

An approximate version of Theorem 4.10 can be stated as follows:

**Theorem 1.1.** *Let  $M$  be a Liouville domain, and let  $\sigma$  be a strongly nondegenerate stop in  $M$ . Let  $\mathcal{B} \subset \mathcal{W}_\sigma(M)$  be the full subcategory of objects supported near  $\sigma$ . Then the inclusion  $\mathcal{W}_\sigma(M) \rightarrow \mathcal{W}(M)$  induces a fully faithful functor*

$$\mathcal{W}_\sigma(M)/\mathcal{B} \rightarrow \mathcal{W}(M),$$

where the quotient is a quotient of an  $A_\infty$  category by a full subcategory in the sense of Lyubashenko-Ovsienko [28].

At the most basic level, Theorem 1.1 shows that the partially wrapped Fukaya category really knows more than the wrapped Fukaya category. That is, restricting to the zero-filtered part doesn't lose information. At an intuitive level, this happens because the "nice" presentation of  $\mathcal{W}(M)$  giving rise to the stop filtration uses a contact form with a large number of canceling Reeb chords. These chords live at different levels in the filtration, so passing to the zero-filtered part results in them no longer canceling. In a more minimal presentation, these chords could be eliminated geometrically.

The key ingredient of the proof of Theorem 1.1 is an auxiliary filtration on  $\mathcal{W}_{\sigma \setminus \{\sigma\}}(M)$ , presented as the trivial quotient  $\mathcal{A} = \mathcal{W}_{\sigma \setminus \{\sigma\}}(M)/\mathcal{B}$ . The benefit of this quotient presentation is that it naturally contains the category  $\mathcal{A}_0 = \mathcal{W}_\sigma(M)/\mathcal{B}(\sigma)$  as the minimally filtered part, which makes it possible to build a homotopy which retracts  $\mathcal{A}$  onto  $\mathcal{A}_0$ . The homotopy itself requires a filtered version of the annulus trick, which was introduced in [2] and extended in [19] and [7]. Specifically, one factors the identity operation as a composition of a product and a coproduct, where the coproduct is required to have one component land in the partially wrapped complex.

## 1.2. Discussion.

**1.2.1. Multiple stops.** In fact, we consider not just a single stop  $\sigma$  but a collection  $\sigma$  of disjoint stops. We refer to the triple  $(M, \lambda, \sigma)$  as a *pumpkin domain* (see Figure 1), and assign to it as above its partially wrapped Fukaya category  $\mathcal{W}_\sigma(M)$ . This is the subcategory of  $\mathcal{W}(M)$  consisting of objects which avoid all the stops in  $\sigma$ , and whose morphism complexes are generated by chords and intersections which avoid all the stops in  $\sigma$ . While this is the same category as  $\mathcal{W}_{\bar{\sigma}}(M)$ , where

$$\bar{\sigma} = \bigcup_{\sigma \in \sigma} \sigma,$$

the equivalent of Theorem 1.1 is more refined. In this case, again assuming that  $\sigma \in \sigma$  is strongly nondegenerate, it states that the canonical functor

$$\mathcal{W}_\sigma(M)/\mathcal{B} \rightarrow \mathcal{W}_{\sigma \setminus \{\sigma\}}(M)$$

is fully faithful.

**1.2.2. Mirror symmetry.** Stops are meant to be a symplectic way of encoding the mirror of an anticanonical divisor. To see how, recall that there are two constructions of the mirror of a toric Fano variety  $X$ . The first [26, 23] produces a Landau-Ginzburg model  $((\mathbb{C}^*)^n, W)$ , where  $W$  is a Laurent polynomial depending on the fan of  $X$ . This comes with a Fukaya-Seidel category, defined in [33] whenever  $W$  is a Lefschetz fibration. Mirror symmetry in various forms has been proved via this approach in [22], [3], and many others.

The second approach is less direct and assigns to  $X$  a singular Lagrangian skeleton  $\Lambda \subset (\mathbb{C}^*)^n$ . The pair  $((\mathbb{C}^*)^n, \Lambda)$  then has two flavors of Fukaya category, called partially wrapped and infinitesimally wrapped, which are meant to be equivalent to the dg-categories  $\text{Coh}(X)$  and  $\text{Perf}(X)$ , respectively. The equivalence proceeds by combining the coherent-constructible correspondence [13] and the Nadler–Zaslow correspondence [32]. This approach is explored by Fang–Liu–Treumann–Zaslow in [14].

In good cases, each of these mirror objects gives rise to a stop in  $M = (\mathbb{C}^*)^n$ . Indeed, if  $W: M \rightarrow \mathbb{C}$  is a superpotential with a compact set of critical values, then the  $W^{-1}(p)$  for  $|p| \gg 1$  projects along the Liouville vector field to a stop  $\sigma_W$ . Similarly, if  $M$  is equipped with a singular Lagrangian skeleton  $\Lambda \subset M$  with  $\partial\Lambda$  a smooth Legendrian, then we may apply the Legendrian neighborhood theorem to thicken  $\partial\Lambda \subset \partial M$  in the contact directions and obtain a stop  $\sigma_\Lambda$ . In the situations where  $W$  and  $\Lambda$  are mirror to smooth toric Fano varieties,  $\partial\Lambda$  is meant to be a Lagrangian skeleton for the generic fiber of  $W$ , so the stops  $\sigma_W$  and  $\sigma_\Lambda$  are isotopic.

Going further, in the Landau-Ginzburg picture, Theorem 1.1 can be thought of as a characterization of the acceleration functor

$$A: \text{Fuk}(M, W) \rightarrow \text{Fuk}(M, 0).$$

This characterization is dual to that of [8] and can be thought of as extending Abouzaid and Seidel’s result to more general Landau-Ginzburg models. In fact, one could dream of a situation in which the theory of pumpkin domains has been extended to intersecting stops. In this case, a theorem analogous to Theorem 1.1 would give a strong refinement of the acceleration functor:

**Conjecture 1.2.** *Suppose  $W = \sum_{i=1}^d W_i$  is a sum of monomials. Then  $\sigma_{W_i}$  and  $\sigma_{W_j}$  are generically expected to intersect. However, if partially wrapped Fukaya categories are developed for intersecting stops, one expects*

$$\mathcal{W}_{\{\sigma_W\}}(M) \cong \mathcal{W}_\sigma(M),$$

where  $\sigma = \{\sigma_{W_1}, \dots, \sigma_{W_d}\}$ . In this case, deleting a stop corresponds to deleting a monomial from  $W$ . For  $(M, W)$  mirror to a toric variety  $X$ , this in turn corresponds to deleting a toric divisor from  $X$ .

**1.2.3. Other frameworks.** When  $M$  is equipped with a Lefschetz fibration  $W$ , one expects that

$$Tw^\pi \text{FS}(W) \cong Tw^\pi \mathcal{W}_{\sigma_W}(M),$$

where  $\text{FS}(W)$  is the Fukaya-Seidel category of the Lefschetz Fibration. The machinery of Fukaya-Seidel categories can be extended at least partially to more general Landau-Ginzburg models [6, 7], in which case one still expects the above to hold when  $W$  has a compact set of critical points. When the critical locus is noncompact, then one can produce two Fukaya-Seidel type categories which differ on whether fiberwise noncompact Lagrangians are allowed as objects. The partially wrapped Fukaya category is meant to be the one in which they are.

When  $M$  is instead equipped with a Lagrangian skeleton  $\Lambda$ , there is a microlocal model for the corresponding partially wrapped Fukaya category due to Nadler [31], defined as the full subcategory of compact objects a large category of microlocal sheaves. It is believed that when  $\partial\Lambda$  is the skeleton

for a stop  $\sigma$ , then Nadler’s category agrees with  $\mathcal{W}_\sigma(M)$ . This equivalence is the main theorem of the upcoming paper of Ganatra, Pardon, and Shende [21].

**1.2.4. Generation.** The primary application of Theorem 1.1 is to give generation statements for partially wrapped Fukaya categories:

**Corollary 1.3.** *Under the assumptions of Theorem 1.1, suppose  $\mathcal{A}_0 \subset \mathcal{W}(M)$  and  $\mathcal{I} \subset \mathcal{B}$  are split-generating full subcategories. Let  $\mathcal{A} \subset \mathcal{W}_\sigma(M)$  be a collection of objects such that  $A(\mathcal{A}) = \mathcal{A}_0$ , where  $A: \mathcal{W}_\sigma(M) \rightarrow \mathcal{W}(M)$  is the inclusion. Then the full subcategory  $\mathcal{A} \cup \mathcal{I}$  of  $\mathcal{W}_\sigma(M)$  split-generates.*

In fact, in future work we will show that the subcategory  $\mathcal{B}$  is the image of an  $A_\infty$  functor  $\iota_\sigma: \mathcal{W}(\sigma) \rightarrow \mathcal{W}_\sigma(M)$ , which is the partially wrapped version of Orlov’s functor

$$\mathrm{Fuk}(F) \rightarrow \mathrm{FS}(W)$$

for a Lefschetz fibration  $W$  with smooth fiber  $F$ . In the mirror picture, this corresponds to the pushforward functor from sheaves on a divisor to sheaves on the total space. The upshot is that a split-generating subcategory  $\mathcal{I}$  can be found as the image under  $\iota_\sigma$  of a split-generating subcategory  $\tilde{\mathcal{I}} \subset \mathcal{W}(\sigma)$ . This reduces the problem of split-generation of partially wrapped Fukaya categories with strongly nondegenerate stops to that of fully wrapped Fukaya categories, for which [20] gives a good answer.

Concretely, consider the Landau-Ginzburg model  $(M, W) = (\mathbb{C}^3, xyz)$ , which is mirror to the pair of pants. This has generic fiber  $(\mathbb{C}^*)^2$ , whose wrapped Fukaya category is generated by the single Lagrangian  $\tilde{L} = (\mathbb{R}_+)^2$ . Because  $\mathcal{B}$  is the image of a functor, we can replace it with a single Lagrangian  $L$  which is the parallel transport of  $\tilde{L}$  over the arc that curves around  $\sigma$ . This expresses the trivial category  $\mathcal{W}(\mathbb{C}^3)$  as a quotient of  $\mathcal{W}_\sigma(M)$  by  $L$ , which means  $L$  generates  $\mathcal{W}_\sigma(M)$ . This result is predicted by mirror symmetry in [6], where Abouzaid and Auroux compute the endomorphism algebra of  $L$  in a “fiberwise wrapped” Fukaya category which is expected to agree with ours.

**1.3. Outline of the paper.** In Section 2, we define stops and pumpkin domains in terms of local models. We prove Proposition 2.6, which justifies that definition, and we use it to give basic examples of pumpkin domains. We then describe how to glue pumpkin domains along stops. This is used in a basic way to control the geometry in Sections 4 and 5. It will also be used in an upcoming paper [35] where we will explain how to recover the partially wrapped Fukaya category of a gluing from the partially wrapped Fukaya categories of the original pumpkin domains and certain functors associated to the stops.

In Section 3, we define partially wrapped Fukaya categories. We then construct continuation functors and homotopies between them, with the objective of proving that partially wrapped Fukaya categories are invariant under isotopies of the stops. The reader who is willing to take this fact for granted may safely skip Sections 3.4 through 3.6.

Section 4 begins by constructing the action on Hochschild homology which goes into the definition of a strongly nondegenerate stop. This allows us to state the precise version of Theorem 1.1. We then introduce a chain-level notion called a *weakly nondegenerate stop* and prove that every strongly nondegenerate stop is weakly nondegenerate (Proposition 4.21). This is the key technical ingredient we use to pass from data in the stop to data on the ambient pumpkin domain.

Finally, in Section 5, we prove the main theorem. We begin by constructing the coproduct operation. Then we construct a sequence of smaller homotopies which interpolate between the composition of product with coproduct and a projection to the partially wrapped part. The key observation here is that every time a long  $X_H$ -chord intersects a stop  $\sigma$ , it does so by first entering a neighborhood of  $\sigma$ , then intersecting  $\sigma$ , and then leaving the neighborhood. Thus, by carefully choosing incidence conditions with the boundary of a neighborhood of  $\sigma$ , we construct in Section 5.7 an operation which looks like the identity but is homotopic to zero.

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## 2. GEOMETRIC SETUP

**2.1. Liouville domains.** Our basic object of study will be Liouville domains  $(M, \lambda_M)$ , which are compact manifolds with boundary such that  $\omega_M := d\lambda_M$  is symplectic, and such that the Liouville vector field  $Z_M$  defined by  $\iota_{Z_M}\omega_M = \lambda_M$  points outward along the boundary. This implies that  $\alpha = \lambda_M|_{\partial M}$  is a contact form, and flowing along  $-Z_M$  gives a collar

$$(U, \lambda_M) \cong ((0, 1] \times \partial M, r\alpha).$$

Attaching the rest of the symplectization of  $\partial M$  gives the **completion**  $\hat{M} := M \cup_{\partial M} [1, \infty) \times \partial M$ , which comes with a natural 1-form  $\hat{\lambda}_M$ , symplectic form  $\hat{\omega}_M$ , and Liouville vector field  $\hat{Z}_M$ .

A good class of mappings between Liouville domains  $F$  and  $M$  is that of **Liouville maps**, which are proper embeddings  $\phi: \hat{F} \hookrightarrow \hat{M}$  such that

$$(2.1) \quad \begin{aligned} \phi^*\hat{\lambda}_M &= \hat{\lambda}_F + df && \text{for some compactly supported } f, \text{ and} \\ \phi_*\hat{Z}_F &= \hat{Z}_M && \text{away from a compact set.} \end{aligned}$$

Note that the second condition is redundant for codimension zero maps. In general, it can be rephrased as saying that the symplectic orthogonal of the image of  $\phi$  lies in the kernel of  $\hat{\lambda}_M$ . A Liouville isomorphism, then, is just a Liouville map that is a diffeomorphism.

A version of Moser's lemma holds in this setting [11]:

**Lemma 2.1.** *Let  $(M, \lambda_M^t)$  be a smooth family of Liouville domains parametrized by  $t \in [0, 1]$ . Suppose  $K \subset \hat{M}$  is a closed subset such that  $\lambda_M^t$  is  $t$ -independent on  $K$ . Then there is a family of Liouville isomorphisms  $\phi_t: (M, \lambda_M^0) \rightarrow (M, \lambda_M^t)$  which is the identity on  $K$ .  $\square$*

**Corollary 2.2.** *Let  $(F, \lambda_F^t)$  and  $(M, \lambda_M^t)$  be smooth families of Liouville domains for  $t \in [0, 1]$ . Suppose there exists a Liouville map  $\phi: (F, \lambda_F^0) \rightarrow (M, \lambda_M^0)$ . Then  $\phi$  extends to an isotopy of Liouville maps  $\phi^t: (F, \lambda_F^t) \rightarrow (M, \lambda_M^t)$ .  $\square$*

Occasionally we will use the stronger notion of an **isomorphism of exact symplectic manifolds**, which is a diffeomorphism  $\phi: M \rightarrow M'$  of exact symplectic manifolds (not necessarily Liouville domains or their completions) satisfying  $\phi^*\lambda_{M'} = \lambda_M$ .

Given two Liouville domains  $M$  and  $M'$ , one can attempt to form their product. The result is an exact symplectic manifold with corners. One can non-canonically round the corners to obtain a Liouville domain. The result completes to  $(\hat{M} \times \hat{M}', \hat{\lambda}_M + \hat{\lambda}_{M'})$ , so the product is at least defined up to isomorphism. We'll use  $M \times M'$  to denote the resulting Liouville domain for any choice of boundary.

If we additionally have a Liouville map  $\phi: F \rightarrow M$ , then the product

$$\phi \times \text{id}_{M'}: F \times M' \rightarrow M \times M'$$

is not quite a Liouville map, but it becomes one if we replace  $\lambda_F$  by  $\phi^*\lambda_M$ . By definition, this doesn't change the Liouville isomorphism class of  $F$ , and in the sequel we'll often make such compactly supported changes implicitly when talking about products.

**2.2. Stops.** Symplectic manifolds often come with additional data, such as a global meromorphic function or a distinguished collection of Lagrangians. For Floer theoretic purposes, this data can often be encoded as a set of framed complex hypersurfaces.

**Definition 2.3.** Let  $(M^{2n}, \lambda_M)$  and  $(F^{2n-2}, \lambda_D)$  be Liouville domains. For  $\rho > 0$ , denote by  $\mathbb{H}_\rho$  the set  $\{z \in \mathbb{C} \mid \Re(z) \geq -\rho\}$  with the standard exact symplectic structure coming from  $\mathbb{C}$ . A **stop** of width  $\rho$  in  $M$  with fiber  $F$  is a proper embedding  $\sigma: \hat{F} \times \mathbb{H}_\rho \rightarrow \hat{M}$  satisfying

$$\sigma^* \hat{\lambda}_M = \hat{\lambda}_F + \lambda_{\mathbb{H}_\rho} + df$$

for some compactly supported  $f$ . If  $\sigma$  is a stop, then  $D_\sigma := \sigma|_{\hat{F} \times \{0\}}$  is a Liouville map, which we'll call its **divisor**. In what follows, we'll often identify  $D_\sigma$  with its image.

The requirement that a stop be a proper map is important. It means that all of the data lives on the boundary, which will be needed to obtain well behaved gluing operations. The notion of width, on the other hand, is just a notational convenience. Specifically, if  $\rho' = t\rho$ , then  $\mathbb{H}_\rho$  and  $\mathbb{H}_{\rho'}$  are isomorphic as exact symplectic manifolds via

$$(x, y) \mapsto (tx, t^{-1}y).$$

We will also sometimes wish to narrow a stop, that is to embed  $\mathbb{H}_\rho$  into some enlarged angular sector

$$S_{\rho,s} = \bar{D}_\rho^2 \cup \left\{ re^{i\theta} \in \mathbb{C} \mid r > 0, |\theta| \leq s \right\}$$

with  $\rho, s > 0$ . While this can't be done in a way that strictly preserves the Liouville form, it can be done in a way that only modifies the Liouville form in some small annulus around zero and fixes the positive real axis. For this, one can take the large time flow of a Hamiltonian which, outside of the annulus, takes the form  $r^2 \sin \theta$ . Crossing with the fiber, this might cause the Liouville vector field to fail to point outward along  $\partial F \times D_\rho^2$ . However, since the modification to  $\lambda_{\mathbb{C}}$  is bounded, we replace  $F$  with a larger piece of  $\hat{F}$  so that  $\lambda_{\mathbb{C}}$  is small compared to  $\lambda_F$ , and hence outward pointingness will be preserved at this new boundary. This shows

**Lemma 2.4.** *Let  $(M, \lambda_M)$  be a Liouville domain and  $\sigma_0: \hat{F} \times S_{\rho,s} \rightarrow \hat{M}$  be a proper codimension zero embedding with*

$$\sigma_0^* \hat{\lambda}_M = \hat{\lambda}_F + \lambda_{S_{\rho,s}} + df$$

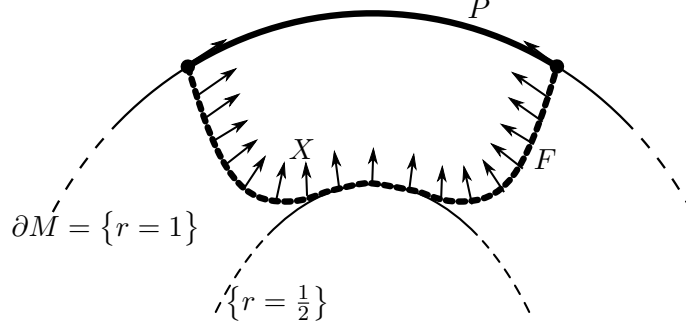
*for some compactly supported  $f$ . Then there is a new Liouville form  $\hat{\lambda}'_M = \hat{\lambda}_M + dg$ , where  $g$  is supported in a small tube around  $\sigma_0(\hat{F} \times \{0\})$ , such that as a map into  $(\hat{M}, \hat{\lambda}'_M)$ ,  $\sigma_0|_{(\hat{F} \times \{0\})}$  extends to a stop  $\sigma$  with  $\sigma(\hat{F} \times \mathbb{R}_+) = \sigma_0(\hat{F} \times \mathbb{R}_+)$ .  $\square$*

**Definition 2.5.** A map satisfying the properties of  $\sigma_0$  above will be called a **narrow stop**.

A stop also constrains the behavior of the Liouville form near its divisor. This too will be needed for gluing, though it is not hard to modify a given Liouville map to look like the divisor of a stop. In fact, we have the following:

**Proposition 2.6.** *Let  $(M^{2n}, \lambda_M)$  be a Liouville domain, and let  $P \subset \partial M$  be a compact hypersurface with boundary such that  $(P, \lambda_M|_P)$  is a Liouville domain. Choose  $f: P \rightarrow [\frac{1}{2}, 1]$  to be a continuous function such that*

- (1)  $f$  is smooth and less than 1 on the interior of  $P$ .
- (2)  $f|_{\partial P} = 1$ .
- (3)  $f^{-1}(r)$  is transversely cut out and contact for  $r > \frac{1}{2}$ .
- (4)  $F = \text{graph}(f) \subset M$  is a smooth submanifold that is parallel to  $Z$  to infinite order along its boundary. See Figure 2.

FIGURE 2. The divisor  $F$  and its framing vector field  $X$ .

Then  $(F, \lambda_M|_F)$  is a Liouville domain, and its inclusion into  $M$  extends to a Liouville map  $\phi$ . Moreover, one can construct a new Liouville form  $\lambda'_M = \lambda_M + dh$  such that, after moving  $\partial M$  out,  $\phi$  becomes the divisor of a stop in  $(M, \lambda'_M)$  with fiber  $F$ .

Note that  $h$  was not required to vanish in a neighborhood of  $\partial M$ . Of course, by Corollary 2.2, one can arrange that it does, at the expense of an isotopy of  $F$ .

*Proof.* To see that  $F$  is a Liouville domain, it suffices to show that  $\omega_M|_F$  is symplectic, since then outward-pointingness is clear from condition 4. In fact, this is automatic near the boundary, since there  $F$  is close to the symplectization of the contact manifold  $\partial P$ . Thus, we can consider only what happens away from the boundary, which allows us to transport the question to  $P$ . For this, let  $\tilde{f}: P \rightarrow M$  be the graph map  $p \mapsto (f, p)$ , so that we are interested in whether  $\tilde{f}^*\omega_M$  is symplectic on the interior of  $P$ . Then  $\tilde{f}^*\lambda_M = f\lambda_M|_P$ , so that  $\tilde{f}^*\omega_M = df \wedge \lambda_M|_P + f\omega_M|_P$ . We compute

$$(\tilde{f}^*\omega_M)^{n-1} = (\omega_M^{n-1})|_P + df \wedge \lambda_M|_P \wedge (\omega_M^{n-2})|_P.$$

The first term is positive because  $(P, \lambda_M|_P)$  is a Liouville domain, while the second term is nonnegative by condition 3. This implies that  $F$  is a Liouville domain, and it follows from the definitions that  $\phi$  is a Liouville map.

Our next step then is to exhibit  $F$  locally as the divisor of a stop. To do this, we will use Moser's argument to modify the Liouville form on  $M$  in a neighborhood of  $F$ . For that to be effective, we will want to frame  $F$  so that, when we try to extend  $\phi$  to a stop, it will know which way points out.

Choose a nonvanishing vector field  $X \in \Gamma(T\hat{M}|_{\hat{F}})$  that is symplectically orthogonal to  $\hat{F}$  and, in the symplectization coordinates  $(r, p)$  on  $(0, \infty) \times \partial M$ , is of the form  $X = (g \frac{\partial}{\partial r}, X_\partial)$ , where  $g \geq 0$  and  $X_\partial$  is tangent to  $P$ . The choice is unique up to scaling by a positive function. Next, pick a second vector field  $Y \in \Gamma(T\hat{M}|_{\hat{F}})$ , also orthogonal to  $\hat{F}$ , such that the radial component of  $Y$  vanishes identically and  $\omega_M(X, Y) = 1$ . By the symplectic neighborhood theorem on a compact part of  $\hat{M}$ , we can find a number  $\rho > 0$  and a symplectic embedding  $\psi: F_2 \times D_\rho^2 \rightarrow \hat{M}$ , where  $F_2$  is the part of  $\hat{F}$  with  $r \leq 2$ , such that

$$(i) \quad \psi|_{F_2 \times \{0\}} = \phi$$

$$(ii) \quad \psi_* \frac{\partial}{\partial x} = X \text{ along } F_2$$

$$(iii) \quad \psi_* \frac{\partial}{\partial y} = Y \text{ along } F_2.$$

where  $x = \Re(z)$  and  $y = \Im(z)$  are the coordinates on  $D_\rho^2$ .

It is time to change  $\lambda$ . Let  $\theta = \lambda_{F_2 \times D_\rho^2} - \psi^*\lambda_M$ . Then  $\theta$  is closed and  $\theta|_{F_2} = 0$ , so we can find a primitive  $h_0$  of  $\theta$  on a neighborhood of  $F_2$  with  $h_0|_{F_2} = 0$ . Shrinking  $\rho$ , we can assume that  $h_0$

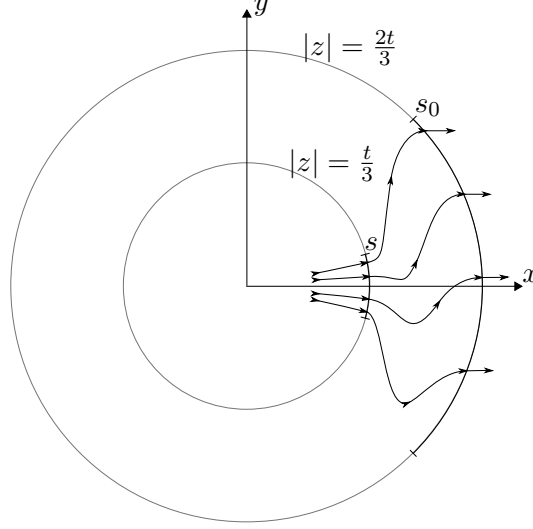


FIGURE 3. One possibility for the image of the narrow stop.

is defined on all of  $F_2 \times D_\rho^2$ . Consider a family of cutoff functions  $\kappa_t: F_2 \times D_\rho^2 \rightarrow [0, 1]$  indexed by  $t \in (0, \rho)$  and satisfying the following conditions:

- (iv)  $\kappa_t$  is independent of the  $F$  component and is rotationally invariant and radially nonincreasing in the  $D^2$  component when  $r \leq 1$
- (v)  $\kappa_t = 1$  when  $|z| \leq \frac{t}{3}$  and  $r \leq 1$
- (vi)  $\kappa_t = 0$  when  $|z| \geq \frac{2t}{3}$  or  $r \geq \frac{3}{2}$
- (vii)  $|d\kappa_t| < \frac{4}{t}$  with respect to some fixed  $t$ -independent product metric on  $F_2 \times D_\rho^2$  which is Euclidean on the  $D_\rho^2$  factor
- (viii)  $\kappa_{t_0}(p, t_0 z) = \kappa_{t_1}(p, t_1 z)$  for all  $(p, z) \in F_2 \times D^2$  and  $t_i \in (0, \rho)$ .

We can rephrase this last condition as saying that shrinking  $t$  corresponds to conjugation by a rescaling of the  $D^2$  component.

We will see that the function  $h$  in the statement of the lemma can be taken to be  $\psi_*(\kappa_t h_0)$  for sufficiently small  $t$ . For now, let us denote that function by  $h_t$ . The first thing to notice is that for  $t$  sufficiently small, the Liouville vector field  $Z_M^t$  associated to  $\lambda_M^t = \lambda_M + dh_t$  points out along the boundary of  $M$ , so that  $(M, \lambda_M^t)$  is a Liouville domain. To see this, note that  $\lambda_M$  vanishes on the symplectic orthogonal to  $[1, 2] \times \partial F$ , where  $[1, 2] \subset (0, \infty)$  is the symplectization component, so  $\theta$  does as well. Thus,  $h_0$  vanishes quadratically on  $[1, 2] \times \partial F$ . This, combined with conditions (vi) and (vii), implies that  $dh_t$  has magnitude  $O(t)$ . Since the condition that  $Z$  points outward is open, this gives the desired conclusion.

It remains to find some  $t$  for which  $\phi$  extends to a stop in  $(M, \lambda_M^t)$ . By Lemma 2.4, it is enough to extend  $\phi$  to a narrow stop. By (v), we can find the disk part of a narrow stop, so we need only find an angular sector over which we can finish extending  $\phi$ . The naive solution here is to just pick a small angular sector and flow out via  $Z$ , which works, but one needs to ensure that this doesn't get snagged on some interesting piece of  $M$ . This is where the framing of  $\psi$  becomes important.

For convenience, let's now identify  $\hat{F} \times D_\rho^2$  with its image under  $\psi$ . Let  $\delta > 0$  be such that if  $r > 1 - \delta$ , then for all sufficiently small  $t$  the flow of  $Z_M^t$  escapes to infinity. Because of (ii), for  $t$



sufficiently small and  $r \leq 1 - \delta$ , the original Liouville vector field  $\hat{Z}_M$  has an  $x$ -component  $(\hat{Z}_M)_x$  which is positive and bounded away from zero along  $\hat{F} \times \{\frac{2t}{3}\}$ . By (vi), the same is true of  $\hat{Z}_M^t$ . By openness, we can find some angle  $s_0 \in (0, \frac{\pi}{2})$  such that  $\hat{Z}_M^t$  points out of  $\hat{F} \times D_{\frac{2t}{3}}^2$  when  $r \leq 1 - \delta$  and the angular  $D^2$ -coordinate  $\theta$  belongs to  $[-s_0, s_0]$ . Since  $(\hat{Z}_M^t)_y$  vanishes on  $F$ ,  $s_0$  can be chosen to be independent of  $t$ . Indeed, as  $t \rightarrow 0$ ,  $s_0$  could be taken to increase to  $\frac{\pi}{2}$ . We want to show that there exists some smaller  $s$  such that  $\hat{Z}_M^t$  takes

$$\hat{F} \times \left\{ \frac{t}{3} e^{i\theta} \mid -s \leq \theta \leq s \right\}$$

through

$$\left( \hat{F} \times \left\{ \frac{2t}{3} e^{i\theta} \mid -s_0 \leq \theta \leq s_0 \right\} \right) \cup (\partial M \times (1 - \delta, \infty)),$$

as in Figure 3.

If we can, then we're done, since we know that the flow of  $\hat{Z}_M^t$  starting anywhere in the latter set escapes to infinity. To accomplish this, it is enough to show that there exist positive constants  $C$  and  $D$ , with  $D$  small (strictly less than  $\frac{c-1}{2c-1-1} \tan s$ ), such that

$$(2.2) \quad \left| \frac{(\hat{Z}_M^t)_y}{(\hat{Z}_M^t)_x} \right| < C \cdot \left| \frac{y}{x} \right| + D \quad \text{and} \quad (\hat{Z}_M^t)_x > 0$$

when  $t$  is small,  $\theta \in [-s_0, s_0]$ , and  $r \leq 1 - \delta$ . It turns out we can take  $C = 1 + \varepsilon$  and  $D = \varepsilon$  for arbitrary small  $\varepsilon > 0$ . Specifically, remember that  $Z_M^t$  is the vector field dual to

$$\lambda_M^t = \kappa_t \cdot \left( \hat{\lambda}_F + \frac{xdy - ydx}{2} \right) + (1 - \kappa_t)\lambda_M + h_0 d\kappa_t.$$

Note we've switched to the interior part of  $M$ , since that is where all our current problems live. Note also that  $|(Z_M^t)_y|$  is given by  $|\lambda_M^t(\frac{\partial}{\partial x})|$  and similarly with  $x$  and  $y$  switched. In the above formula, the  $\hat{\lambda}_F$  term doesn't affect (2.2), so it can be ignored. Furthermore, to lowest order, the  $h_0 d\kappa_t$  term is strictly beneficial from the perspective of (2.2). To see this, note that there is a positive function  $E \in C^\infty(F)$  such that  $dh_0 = -E dy + O(|z|)$  near  $F \times \{0\}$ , so that  $h_0 = -Ey + O(|z|^2)$ . Thus, using conditions (vii), we see that

$$h_0 d\kappa_t = -E \frac{xy}{|z|} \frac{\partial \kappa_t}{\partial |z|} dx - E \frac{y^2}{|z|} \frac{\partial \kappa_t}{\partial |z|} dy + O(t).$$

Now  $\frac{\partial \kappa_t}{\partial |z|}$  is nonpositive by assumption, and  $dx$  is dual to  $-\frac{\partial}{\partial y}$ , so the first term leads to a negative  $y$ -component when  $y$  is positive and a positive  $y$ -component when  $y$  is negative. This means that its contribution to  $Z_M^t$  will never enlarge  $|\theta|$ . Similarly, the second term is dual to a nonnegative function times  $\frac{\partial}{\partial x}$ , so it too will never enlarge  $|\theta|$ . Hence, it suffices to show that (2.2) can be satisfied with  $h_0$  replaced by  $\tilde{h}_0 = h_0 + Ey$ , i.e. after discarding the leading order term.

For this, we will need to separately consider two pieces. Pick  $w \in (\frac{1}{3}, \frac{2}{3})$  to be such that  $\kappa_t(p, wt) \neq 1$  and, if  $|z| < wt$ ,  $\theta \in [-s_0, s_0]$ , and  $r \leq 1 - \delta$ , then

$$(2.3) \quad \|\tilde{h}_0 d\kappa_t(p, z)\| < \frac{\varepsilon \kappa_t}{4} x$$

$$(2.4) \quad (1 - \kappa_t(p, wt)) \cdot \|d(\lambda_M(\frac{\partial}{\partial x}))(p, 0)\| < \frac{\varepsilon \kappa_t}{6}.$$

When  $|z| \geq wt$ , there is a positive  $t$ -independent lower bound on  $(1 - \kappa_t)\lambda_M(\frac{\partial}{\partial y})$ , whereas after replacing  $h_0$  by  $\tilde{h}_0$  every other term of  $\lambda_M^t(\frac{\partial}{\partial x})$  and  $\lambda_M^t(\frac{\partial}{\partial y})$  tends uniformly to zero as  $t \rightarrow 0$ . Thus,

we can satisfy (2.2) as long as we can reach  $|z| = wt$ . But when  $|z| < wt$ , we can again shrink  $t$  so that (2.4) implies

$$(2.5) \quad (1 - \kappa_t) |\lambda_M(\frac{\partial}{\partial x})| < \frac{\varepsilon \kappa_t}{5} x.$$

But now, using the positivity of  $\lambda_M(\frac{\partial}{\partial y})$ , we have

$$\begin{aligned} \left| \frac{(\hat{Z}_M^t)_y}{(\hat{Z}_M^t)_x} \right| &= \left| \frac{\frac{1}{2}\kappa_t y - (1 - \kappa_t)\lambda_M(\frac{\partial}{\partial x}) - \tilde{h}_0 d\kappa_t(\frac{\partial}{\partial x})}{\frac{1}{2}\kappa_t x + (1 - \kappa_t)\lambda_M(\frac{\partial}{\partial y}) + \tilde{h}_0 d\kappa_t(\frac{\partial}{\partial y})} \right| \\ &\leq \frac{|y| + \frac{2}{5}\varepsilon x + \frac{1}{2}\varepsilon x}{x - \frac{1}{2}\varepsilon x} \\ &< (1 + \varepsilon) \cdot \left| \frac{y}{x} \right| + \varepsilon \end{aligned}$$

as desired.  $\square$

*Example 2.7.* Let  $W: M \rightarrow \mathbb{C}$  be an exact Lefschetz fibration [33], and let  $\gamma: [0, \infty) \rightarrow \mathbb{C}$  be a properly embedded, asymptotically radial ray that avoids the critical values of  $W$ . Then we can modify the Liouville structure on  $M$  in a neighborhood of  $W^{-1}(\gamma)$  to obtain a stop modeled on  $W^{-1}(\gamma(0))$ .

More generally, we can do the above for any holomorphic fibration. This is the construction that we will use to define the partially wrapped Fukaya category of a Landau-Ginzburg model.

*Example 2.8.* Let  $\Lambda \subset \partial M$  be a smooth closed Legendrian submanifold. By the Legendrian neighborhood theorem,  $\Lambda$  has a tubular neighborhood  $U \subset \partial M$  which is isomorphic as a contact manifold with 1-form to a neighborhood of the zero section in the 1-jet bundle  $J^1\Lambda$ . Identifying  $U$  with that neighborhood, the submanifold

$$T^*\Lambda \cap U \subset U \subset M$$

is then a Liouville hypersurface of  $\partial M$ , and applying Proposition 2.6 gives a stop  $\sigma_\Lambda$ .

### 2.3. Pumpkin domains.

**Definition 2.9.** From here on, the basic geometric object we will deal with is a **Liouville domain with disjoint stops**, or **pumpkin domain** for short. This is a triple  $(M, \lambda_M, \sigma)$  with  $(M, \lambda_M)$  a Liouville domain and  $\sigma = \{\sigma_1, \dots, \sigma_k\}$  a collection of stops in  $M$  such that the images of  $\sigma_i$  and  $\sigma_j$  are disjoint for  $i \neq j$ . For technical simplicity, we make the additional assumption that every stop  $\sigma_i$  strictly preserves the Liouville form on  $\hat{M} \setminus M$ . Since a pumpkin domain only has finitely many stops, this can always be achieved by moving  $\partial M$  out.

An **equivalence** of pumpkin domains  $(M, \lambda_M, \sigma)$  and  $(M', \lambda_{M'}, \sigma')$  is a homotopy of collections of disjoint stops  $\sigma^t = \{\sigma_1^t, \dots, \sigma_k^t\}$ , where  $\sigma_i^t$  is a stop in  $M$  with fiber  $(F_i, \lambda_{F_i}^t)$ , together with a Liouville isomorphism  $\psi: M \rightarrow M'$  such that  $\sigma^0 = \sigma$  and  $\psi \circ \sigma_i^1 = \sigma'_i$ . Here,  $M$  and  $M'$  are required to have the same number of stops.

We'll usually abuse notation and use  $M$  to refer to the pumpkin domain  $(M, \lambda_M, \sigma)$ .

*Example 2.10.* Fix a positive integer  $n$ , and consider the map  $u_n: \mathbb{C} \rightarrow \mathbb{C}$  given by

$$u_n(z) = z^{n+1} - 1.$$

Then  $u^*\lambda_{\mathbb{C}}$  is almost a Liouville form on  $\mathbb{C}$ , except that its derivative vanishes at the origin. Choose a cutoff function  $\kappa: \mathbb{R} \rightarrow [0, 1]$  with  $\kappa(x) = 1$  for  $x \leq \frac{1}{4}$  and  $\kappa(x) = 0$  for  $x \geq \frac{1}{2}$ . Next, choose  $\epsilon > 0$  such that  $u^*\omega_{\mathbb{C}} + \epsilon d(\kappa(|z|)\lambda_{\mathbb{C}})$  is symplectic. Let  $\sigma^n = \{\sigma_0^n, \dots, \sigma_n^n\}$  be the set of  $u_n$ -lifts of

the inclusion  $\mathbb{H}_{\frac{1}{2}} \hookrightarrow \mathbb{C}$  ordered counterclockwise with  $\sigma_0^n$  specified by  $\sigma_0^n(0) = 1$ . Then this data describes a pumpkin domain

$$\mathbb{C}_n := (\mathbb{C}, u^* \lambda_{\mathbb{C}} + \epsilon \kappa(|z|) \lambda_{\mathbb{C}}, \sigma^n).$$

Its underlying Liouville domain is Liouville isomorphic to  $\mathbb{C}$ , and it has  $n + 1$  stops, all with fiber the point. Note that  $i\mathbb{R} \subset \mathbb{C}_1$  is invariant under the flow of the Liouville vector field.

**Definition 2.11.** Let  $M$  be a Liouville domain. Then the *stabilization* of  $M$  is the pumpkin domain

$$\Sigma M = M \times \mathbb{C}_1.$$

As a Liouville domain,  $\Sigma M$  is just isomorphic to the product  $M \times \mathbb{C}$ . As for the stops, there are two of them, both with fiber  $M$ , and their divisors sit over 1 and  $-1$ .

*Remark 2.12.* Though we will not deal with them, one is sometimes given manifolds with stops that intersect. In this situation, it is reasonable to ask that the stops are **orthogonal**: it should be the case that if  $\sigma_i$  has fiber  $F_i$  and width  $\rho_i$ , then there is a Liouville splitting

$$\text{image}(\sigma_1) \cap \text{image}(\sigma_2) = (D_{\sigma_1} \cap D_{\sigma_2}) \times \mathbb{H}_{\rho_1} \times \mathbb{H}_{\rho_2}$$

that induces the splittings given by each of the stops individually. This gives rise to a more natural setting of Liouville domains with stops, not necessarily disjoint, and here the fiber of a stop will again be a Liouville domain with stops. Well definedness is achieved by induction on dimension. This approach has the advantage of being closed under products; in particular it admits arbitrary stabilizations.

**2.4. Hamiltonians for stops.** Let  $M$  be a pumpkin domain. To obtain an invariant Floer theory, we will need to find a class of Hamiltonians on  $M$  that is well adapted to the pumpkin structure. To state a compatibility condition, we need a convention for Hamiltonian vector fields, which we set as  $dH = -\iota_{X_H} \omega$ .

**Definition 2.13.** A **compatible Hamiltonian** on  $(M, \lambda_M, \sigma)$  is a function  $H \in C^\infty(\hat{M})$  such that

- (1)  $H$  is strictly positive.
- (2)  $dH(\hat{Z}_M) = 2H$  outside of a compact set.
- (3)  $X_H$  is tangent to  $D_\sigma$  for each stop  $\sigma \in \sigma$ .
- (4) For each stop  $\sigma \in \sigma$ ,  $d\theta(X_H)$  is nowhere negative on a neighborhood of  $\sigma(\hat{F} \times \mathbb{R}_+)$ . Here,  $\theta$  is the angular coordinate on the right half plane.

In particular, this last condition says that any integral curve for  $X_H$  has only positive intersections with  $\sigma(\hat{F} \times \mathbb{R}_+)$ .

It's worth noting that the space of Hamiltonians compatible with a given pumpkin domain forms a convex cone. Additionally, if one thinks of a Liouville domain  $M$  as a pumpkin domain with no stops, then a compatible Hamiltonian on  $M$  is just a positive quadratic Hamiltonian, as usual.

**Lemma 2.14.** *Every pumpkin domain admits a compatible Hamiltonian.*

*Proof.* We need to show that conditions (3) and (4) can be achieved. For this, let  $(M, \lambda_M, \sigma)$  be a pumpkin domain, and assume without loss of generality that  $\sigma$  has only one element  $\sigma$  with fiber  $F$  and width  $\rho$ . Fix a compatible Hamiltonian  $g$  on  $F$ . We want to extend this compatibly to all of  $\hat{F} \times \mathbb{H}_\rho$ , since then we can just patch it into  $M$ . For that, choose a nondecreasing smooth function  $a: \mathbb{R}_{\geq 0} \rightarrow [0, 1]$  with  $a|_{[0,1]} = 0$  and  $a|_{[2,\infty]} = 1$ , and set  $f(z) = |z|^4 a(|z|^4)$  as a function on  $\mathbb{H}_\rho$ . Define  $h: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  by  $h(x) = \frac{1}{2} a(x) \log x$ . Our candidate Hamiltonian is given by

$$(2.6) \quad H_{\text{cand}}(p, z) = e^{2h(f(z))} g(\phi_F(-h(f(z)), p)) + e^{2h(g(p))} f(\phi_{\mathbb{C}}(-h(g(p)), z))$$

where  $\phi_F(t, \cdot)$  and  $\phi_{\mathbb{C}}(t, \cdot)$  are the time  $t$  flows of the Liouville vector fields of  $F$  and  $\mathbb{C}$ , respectively. One readily checks that  $H_{\text{cand}}$  satisfies conditions (1)-(3), so we need only to find some condition under which it satisfies (4). Now, since  $f$  is rotationally invariant, condition (4) is equivalent to the requirement that  $\frac{\partial}{\partial x} H_{\text{cand}} \geq 0$  for  $z = x \in \mathbb{R}_+$ . This clearly holds for the second term, and for the first we compute

$$\frac{\partial}{\partial x} e^{2h \circ f} g(\phi_F(-h(f(x)), p)) = e^{2h \circ f} (h \circ f)'(x) \cdot (2g - dg(\hat{Z}_F)).$$

Since  $h \circ f$  is nonnegative and nondecreasing, it is enough to require  $dg(\hat{Z}_F) \leq 2g$  globally. This can be achieved by just making  $g$  bigger on the interior of  $F$ .  $\square$

*Remark 2.15.* We will usually have not one, but a family of compatible Hamiltonians parametrized by some space  $\Sigma$ . In this situation, we require that the compact set in condition 2 in Definition (2.13) can be chosen  $\Sigma$ -independently.

There is another class of Hamiltonians we will want to consider, namely those whose vector fields generate Liouville automorphisms. These have a characterization similar to that of compatible Hamiltonians:

**Definition 2.16.** A **linear Hamiltonian** on  $(M, \lambda_M, \sigma)$  is a function  $H^\ell \in C^\infty(\hat{M})$  such that

- (1)  $dH^\ell(\hat{Z}_M) = H^\ell$  outside of a compact set.
- (2)  $X_{H^\ell}$  is tangent to  $D_\sigma$  for each stop  $\sigma \in \sigma$ .

A linear Hamiltonian is said to be **transverse** if, outside of a compact piece of  $\hat{M}$ , it is of the form  $b\sqrt{H}$  for  $b \in \mathbb{R}$  and  $H$  a compatible Hamiltonian. Near infinity, this is equivalent to asking that its flow is either identically zero or transverse to the contact distribution, hence the name.

**Lemma 2.17.** *Let  $\psi_t$  be the flow of a time-dependent linear Hamiltonian. Then outside of a compact set,  $\psi_t$  can be approximated rel endpoints in  $C^0$  by the flow of a time-dependent transverse Hamiltonian.*

*Proof.* Fix an auxiliary compatible Hamiltonian  $H$ , where  $H$  is of the form 2.6 with  $f$  interpolating from  $|z|^2$  to  $|z|^4$  instead of from 0 to  $|z|^4$ . Write  $H^\ell$  for the positive transverse Hamiltonian  $\sqrt{H}$  and let  $\phi_{H^\ell}^\tau$  be its time  $\tau$  flow. For sufficiently large  $A \in \mathbb{R}$ , note that  $\phi_{H^\ell}^{At} \circ \psi_t$  is the flow of a transverse Hamiltonian, and fix such an  $A$ . Then for  $f: [0, 1] \rightarrow [0, 1]$  a nondecreasing function of slope at most 2, we have that  $\psi_t^f = \phi_{H^\ell}^{2At} \circ \psi_{f(t)}$  is also the flow of a transverse Hamiltonian. Pick  $f$  to be as above, constant on many small intervals, and  $C^0$ -close to  $\text{id}_{[0,1]}$  rel endpoints. Then we can compose  $\psi_t^f$  with a large flow in the direction of  $-X_{H^\ell}$  on the constant intervals to obtain the desired approximation.  $\square$

**2.5. Geometric gluing.** Let  $(M, \lambda_M, \sigma)$  and  $(M', \lambda_{M'}, \sigma')$  be pumpkin domains. Let  $F_i$  and  $F'_j$  be the fibers of  $\sigma_i$  and  $\sigma'_j$ , respectively. When  $F_i$  and  $F'_j$  are isomorphic, we would like to form a new pumpkin domain  $M_{\sigma_i \# \sigma'_j} M'$ . To do this, let us fix an isomorphism  $\phi: F_i \rightarrow F'_j$ . Replacing the 1-form  $\lambda_{F'_j}$  by  $(\phi^{-1})^* \lambda_{F_i}$ , we can assume  $\phi$  is an isomorphism of exact symplectic manifolds. Due to the noncompact  $\mathbb{H}_\rho$  factor in the domain of  $\sigma'_j$ , this causes  $\sigma'_j$  to cease being a stop. To fix that, we need to modify  $\lambda_{M'}$ . Now  $\sigma_i$  and  $\sigma'_j$  are stops with the *same* fiber  $F$ , and so we can make one last modification of  $\lambda_M$  and  $\lambda_{M'}$ , this one compactly supported and exact, to assume that  $\sigma_i$  and  $\sigma'_j$  themselves strictly preserve the Liouville forms.

That done, we can write down the gluing. Pick a positive number  $a$  that is smaller than the widths  $\rho_i$  and  $\rho'_j$  of  $\sigma_i$  and  $\sigma'_j$ . With this data, we can define the underlying Liouville domain of  $M_{\sigma_i \# \sigma'_j} M'$  as

$$(\hat{M} \setminus \sigma_i(\hat{F} \times \{\Re(z) \geq a\})) \amalg (\hat{M}' \setminus \sigma'_j(\hat{F} \times \{\Re(z) \geq a\})) / \sim$$

where  $\sim$  is the identification

$$\sigma_i(\hat{F} \times \{-a < \Re(z) < a\}) = \sigma'_j(\hat{F} \times \{-a < \Re(z) < a\})$$

via  $(p, z) \mapsto (p, -z)$ . The stops are just

$$\sigma_{M_{\sigma_i} \#_{\sigma'_j} M'} := (\sigma \setminus \{\sigma_i\}) \amalg (\sigma' \setminus \{\sigma'_j\}),$$

which makes sense since the stops are disjoint.

Suppose now that we had a 1-parameter family of pumpkin domains  $(M, \lambda_M, \sigma^t)$ , that is an equivalence between  $(M, \lambda_M, \sigma^0)$  and  $(M, \lambda_M, \sigma^1)$ . Then the diffeomorphism type of  $M_{\sigma_i} \#_{\sigma'_j} M'$  is independent of  $t$ , and by Moser's lemma we get a family of Liouville isomorphisms

$$\Psi_t: M_{\sigma_i^0} \#_{\sigma'_j} M' \rightarrow M_{\sigma_i^1} \#_{\sigma'_j} M'.$$

Pulling back the stops in  $M_{\sigma_i} \#_{\sigma'_j} M'$  via  $\Psi_t$ , we see that our homotopy of stops in  $M$  results only in a homotopy of stops in the gluing. Repeating this on the  $M'$  side, we obtain

**Lemma 2.18.** *Gluing descends to an operation on equivalence classes of pumpkin domains, and at this level it depends only on the triple  $(\sigma_i, \sigma'_j, [\phi])$ . Here,  $[\phi]$  is the connected component that  $\phi$  belongs to in the space of Liouville isomorphisms from  $F_i$  to  $F'_j$ .*  $\square$

We will often find that the image of a stop is too small to contain interesting global geometric objects. To remedy this, we will often make use of the following construction.

**Definition 2.19.** Let  $(M, \lambda_M, \sigma)$  be a pumpkin domain, and let  $\sigma \in \sigma$  be a stop with fiber  $F$ . Then the **trivial gluing** at  $\sigma$ , written  $M[\sigma]$ , is the pumpkin domain  $M_{\sigma} \#_{\sigma_0} \Sigma F$ .

Trivial gluing effectively replaces  $\sigma$  with  $\sigma_1$  and doesn't change the pumpkin equivalence class of  $M$ . Indeed, it can be achieved by homotoping  $\lambda_M$  in the class of Liouville forms and moving  $\sigma$  out. The benefit of trivial gluing is that it gives rise to the  $Z_M$ -invariant hypersurface  $\hat{F} \times i\mathbb{R}$ .

### 3. PARTIALLY WRAPPED FUKAYA CATEGORIES

**3.1. Lagrangian Floer cohomology.** For convenience of notation, we'll assume everything in sight is graded. Specifically, we require that all of our Liouville domains satisfy  $2c_1(M) = 0$ , and further that they come with a choice of fiberwise universal cover  $\widetilde{LGr}(M)$  of their bundle of unoriented Lagrangian Grassmannians. Given two Liouville domains  $M_1$  and  $M_2$ , their product is graded in the unique way that extends  $\widetilde{LGr}(M_1) \times_{\mathbb{Z}} \widetilde{LGr}(M_2)$ . All codimension zero symplectic embeddings will be assumed to preserve these covers.

**Definition 3.1.** Given a pumpkin domain  $M$ , a **Lagrangian**  $L \subset M$  is an exact, oriented, properly embedded Lagrangian submanifold of  $\hat{M}$  which is parallel to  $\hat{Z}_M$  outside of a compact set. It is required to be graded in the standard sense, namely that it is equipped with a lift to  $\widetilde{LGr}(M)$  of the natural section  $L \rightarrow LGr(M)$ . For compatibility with the pumpkin structure, we require that  $L$  does not intersect any  $\sigma_i(\hat{F} \times \mathbb{R}_{\geq 0})$ .

An **interior Lagrangian** is a Lagrangian which completely avoids the images of the stops and whose image under the projection  $\hat{M} \setminus M \rightarrow \partial M$  does as well. It is easy to see that any Lagrangian is isotopic via a linear Hamiltonian to an interior Lagrangian.

Given a compatible Hamiltonian  $H$  on  $M$ , we want to consider a class  $\mathcal{J}(M, H)$  of almost complex structures which are adapted to  $H$ . An element  $J \in \mathcal{J}(M, H)$  is a smooth almost complex structure on  $\hat{M}$  which is compatible with  $\hat{\omega}_M$  and satisfies the following three conditions. First, there is some  $c > 0$  such that

$$(3.1) \quad dH \circ J = -cH\hat{\lambda}_M$$

outside of a compact set. Second, the restriction  $J|_{\ker dH \cap \ker \hat{\lambda}_M}$ , i.e. the contact portion of  $J$ , is asymptotically  $\hat{Z}_M$ -invariant. Third, for each stop  $\sigma \in \boldsymbol{\sigma}$ , we require that the projection to  $\mathbb{H}_\rho$  is holomorphic along  $D_\sigma$ . In other words, the divisor of each stop is required to be an almost complex submanifold, and the restriction of  $J$  to its symplectic orthogonal coincides with multiplication by  $i$  in the base.

**Lemma 3.2.** *For any pumpkin domain  $(M, \lambda_M, \boldsymbol{\sigma})$  and compatible Hamiltonian  $H$ , the space  $\mathcal{J}(M, H)$  is contractible and non-empty.*

*Proof.* We prove only the last part, for which it is enough to construct such an almost complex structure near the divisor of a stop  $\sigma$ . Let  $F$  be the fiber of  $\sigma$ , and pick an almost complex structure  $J_F \in \mathcal{J}(F, H|_F)$ . Since the symplectic orthogonal to  $D_\sigma$  lies in the kernel of both  $dH$  and  $\lambda_M$  outside of a compact set, there is no obstruction to extending  $J_F$  to  $T\hat{M}|_{D_\sigma}$  while satisfying Equation (3.1). Now just extend to the rest of  $\hat{M}$ .  $\square$

To endow  $\mathcal{J}(M, H)$  with the structure of a complete metric space, one needs to fix the compact set for (3.1). This prevents the existence of a sequence of almost complex structures which satisfies (3.1) only outside of ever larger compact sets, so that the limit satisfies it nowhere. To obtain transversality results, we will require that  $H$  is quadratic and all Lagrangians are  $\hat{Z}_M$ -invariant outside the compact set. We choose the compact sets implicitly as part of the data of  $H$ , for example to equal  $H^{-1}((-\infty, r+1])$ , where  $H^{-1}((-\infty, r])$  is the smallest sublevel set of  $H$  outside of which it is strictly quadratic and the Lagrangians are strictly conical.

We will in fact need time-dependent, or more generally domain-dependent almost complex structures. For this, suppose  $\Sigma$  is a smooth manifold, possibly with boundary or corners, and that we've chosen a  $\Sigma$ -parametrized family of compatible Hamiltonians  $H$ . Denote by  $\mathcal{J}^\Sigma(M, H)$  the set of smooth maps  $J: \Sigma \rightarrow \mathcal{J}(\hat{M}, \hat{\omega}_M)$  satisfying

$$J(z) \in \mathcal{J}(M, H(z))$$

for all  $z \in \Sigma$ , and such that (3.1) holds pointwise outside of a  $\Sigma$ -independent compact subset of  $\hat{M}$ . Here,  $\mathcal{J}(\hat{M}, \hat{\omega}_M)$  is the space of all  $\hat{\omega}_M$ -compatible almost complex structures. Likewise, for families of domain-dependent almost complex structures, we require that the compact set can be chosen uniformly for the family. In practice, we will choose the compact set implicitly to be a sublevel set for the family of Hamiltonians.

Now suppose  $L_0$  and  $L_1$  are Lagrangians in  $M$ , and  $H$  is a compatible Hamiltonian. We say that  $H$  is **nondegenerate** for the pair  $(L_0, L_1)$  if the following two conditions hold. First,  $\phi(L_0)$  is transverse to  $L_1$ , where  $\phi$  is the time 1 flow of  $X_H$ . Second, if  $\dim(M) \geq 4$ , then no two points of  $\phi(L_0) \cap L_1$  are Liouville translates of one another. If  $H$  is nondegenerate, set  $\mathcal{X}(L_0, L_1; H)$  to be the set of time 1  $X_H$ -chords starting on  $L_0$  and ending on  $L_1$ . Since everything was graded, chords  $\gamma \in \mathcal{X}(L_0, L_1)$  are equipped with a degree  $\deg(\gamma)$  given by topological intersection number with the Maslov cycle.

For  $J \in \mathcal{J}^{[0,1]}(M, H)$ , we consider maps

$$u: Z = \mathbb{R} \times [0, 1] \rightarrow \hat{M}$$

mapping  $\mathbb{R} \times \{0\}$  to  $L_0$  and  $\mathbb{R} \times \{1\}$  to  $L_1$ . For fixed  $\gamma_+$  and  $\gamma_-$  in  $\mathcal{X}(L_0, L_1, H)$ , let  $\tilde{\mathcal{R}}(\gamma_+; \gamma_-)$  be the collection of such maps satisfying Floer's equation

$$(3.2) \quad \partial_s u + J(t)(\partial_t u - X_H) = 0$$

with  $s$  and  $t$  the coordinates on  $\mathbb{R}$  and  $[0, 1]$ , respectively, and such that

$$(3.3) \quad \lim_{s \rightarrow \pm\infty} u(s, \cdot) = \gamma_\pm.$$

The transversality arguments in [17] show that

**Lemma 3.3.** *Fix a pumpkin domain  $(M, \lambda_M, \sigma)$ , Lagrangians  $L_0$  and  $L_1$ , and a nondegenerate compatible Hamiltonian  $H$ . Then there is a comeager subset*

$$\mathcal{J}_{\text{reg}}^{[0,1]}(M, H) \subset \mathcal{J}^{[0,1]}(M, H)$$

*such that, for any  $J \in \mathcal{J}_{\text{reg}}^{[0,1]}(M, H)$  and  $\gamma_{\pm} \in \mathcal{X}(L_0, L_1)$ ,  $\tilde{\mathcal{R}}(\gamma_+; \gamma_-)$  is a smooth manifold of dimension  $\deg(\gamma_-) - \deg(\gamma_+)$ . In this case, the translation  $\mathbb{R}$ -action on  $\tilde{\mathcal{R}}(\gamma_+; \gamma_-)$  is free if and only if  $\gamma_+ \neq \gamma_-$ .  $\square$*

In the above situation, if  $\deg(\gamma_-) > \deg(\gamma_+)$ , define  $\mathcal{R}(\gamma_+; \gamma_-) := \tilde{\mathcal{R}}(\gamma_+; \gamma_-)/\mathbb{R}$ . This is again a smooth manifold, and it has a Gromov compactification  $\overline{\mathcal{R}}(\gamma_+; \gamma_-)$  given by adding broken Floer trajectories. To define the Floer complex, we need one more standard result, first proven in [15].

**Lemma 3.4.** *Suppose we are in the situation of Lemma 3.3, and that  $J \in \mathcal{J}_{\text{reg}}^{[0,1]}(M, H)$ . Then the following hold.*

- (1) *If  $\deg(\gamma_-) - \deg(\gamma_+) = 1$ , then  $\overline{\mathcal{R}}(\gamma_+; \gamma_-) = \mathcal{R}(\gamma_+; \gamma_-)$  is a finite set.*
- (2) *If  $\deg(\gamma_-) - \deg(\gamma_+) = 2$ , then  $\overline{\mathcal{R}}(\gamma_+; \gamma_-) = \mathcal{R}(\gamma_+; \gamma_-) \amalg \mathcal{R}_{(1)}(\gamma_+; \gamma_-)$  is a compact 1-manifold with boundary, where*

$$\mathcal{R}_{(1)}(\gamma_+; \gamma_-) := \coprod_{\substack{\tilde{\gamma} \in \mathcal{X}(L_0, L_1) \\ \deg(\tilde{\gamma}) = \deg(\gamma_+) + 1}} (\mathcal{R}(\gamma_+; \tilde{\gamma}) \times \mathcal{R}(\tilde{\gamma}; \gamma_-)).$$

*In this case, its boundary is precisely  $\mathcal{R}_{(1)}(\gamma_+; \gamma_-)$ .*

*Note that the last sum is finite for action reasons.  $\square$*

Let  $\mathbb{K}$  be a field of characteristic 2. We define a graded vector space  $CW^*(L_0, L_1)$  by degree as

$$CW^k(L_0, L_1) = \bigoplus_{\substack{\gamma \in \mathcal{X}(L_0, L_1) \\ \deg(\gamma) = k}} \mathbb{K}\gamma.$$

Fixing  $J \in \mathcal{J}_{\text{reg}}^{[0,1]}(M, H)$ , we define a differential

$$\delta: CW^k(L_0, L_1) \rightarrow CW^{k+1}(L_0, L_1)$$

by

$$\delta\gamma_+ = \sum_{\deg(\gamma_-) - \deg(\gamma_+) = 1} \#\mathcal{R}(\gamma_+; \gamma_-) \cdot \gamma_-,$$

where  $\#\mathcal{R}(\gamma_+; \gamma_-)$  is the mod-2 count of elements of  $\mathcal{R}(\gamma_+; \gamma_-)$ . Now,  $\delta^2$  counts broken trajectories connecting chords of index difference 2, which are precisely elements of some  $\mathcal{R}_{(1)}$ . By Lemma 3.4, this makes up the boundary of some one-dimensional moduli space, so it has an even number of elements. This means  $\delta^2 = 0$ , so  $(CW^*(L_0, L_1), \delta)$  is a cochain complex, called the **wrapped Floer cochain complex** of  $L_0$  with  $L_1$ .

Each stop  $\sigma \in \sigma$  induces a filtration by  $\mathbb{N}$  on  $CW^*(L_0, L_1)$  as follows: Condition (4) in Definition 2.13 means that for any  $\gamma \in \mathcal{X}(L_0, L_1)$ , the intersections of  $\gamma$  with  $\sigma(\hat{F} \times \mathbb{R}_+)$  are all positive. Denote the number of such intersections  $n_{\sigma}(\gamma)$ .

**Lemma 3.5.** *The Floer differential  $\delta$  never increases  $n_{\sigma}$ . In other words,  $n_{\sigma}$  induces a filtration on wrapped Floer cochain complexes.*

*Proof.* Suppose  $u \in \mathcal{R}(\gamma_+; \gamma_-)$ . Since our Lagrangians avoid  $\sigma(\hat{F} \times \mathbb{R}_{\geq 0})$ , the winding number of  $\partial u$  about  $D_{\sigma}$  coincides with the difference  $n_{\sigma}(\gamma_+) - n_{\sigma}(\gamma_-)$ . By definition, this winding number also gives the topological intersection number of  $u$  with  $D_{\sigma}$ . Thus, it is enough to show that  $u$  has only positive intersections with  $D_{\sigma}$ .

Recall Gromov's trick, which interprets  $H$ -perturbed holomorphic curves as unperturbed holomorphic sections of  $Z \times \hat{M}$ , for a special choice of almost complex structure. Since  $J_t$  fixes  $D_\sigma$  and  $X_H$  is tangent to  $D_\sigma$ , Gromov's trick will present  $Z \times D_\sigma$  as an almost complex submanifold of  $Z \times \hat{M}$ . This means its intersections with the section given by  $u$  are all positive, and linear algebra shows that the same holds for the original intersections.  $\square$

Combining the above for all the stops  $\sigma_i \in \sigma$ , we get a filtration on  $CW^*(L_0, L_1)$  by  $\mathbb{N}^{|\sigma|}$ .

**Definition 3.6.** The **partially wrapped Floer cochain complex** of  $L_0$  with  $L_1$ , denoted  $CW_\sigma^*(L_0, L_1)$ , is the 0-filtered part of  $CW^*(L_0, L_1)$ . In other words, it is the subcomplex generated by those  $H$ -chords which don't traverse any of the stops.

**3.2.  $A_\infty$  categories.** We'll now construct the Fukaya  $A_\infty$ -categories that enhance the above Floer complexes. To begin, we establish some notation for associahedra and strip-like ends.

Following [33], for  $d \geq 2$ , let  $\mathcal{R}^{d+1}$  denote the space of disks with  $d+1$  boundary punctures, labeled  $\zeta_0$  to  $\zeta_d$  and ordered counterclockwise, modulo conformal equivalence.  $\mathcal{R}^{d+1}$  lives naturally as interior of the  $d$ 'th Stasheff associahedron  $\bar{\mathcal{R}}^{d+1}$ , where the boundary faces are products of lower dimensional associahedra indexed by irreducible rooted trees with  $d$  ordered leaves. To be explicit, by irreducible we mean that the root vertex has valency at least two and the internal vertices have valency at least three.

Associated to the associahedra are their dg-operad of top cells, and an  $A_\infty$ -category is a category over this operad. Explicitly, an  $A_\infty$ -category  $\mathcal{A}$  consists of

- (1) A collection of objects  $\text{Ob}\mathcal{A}$ .
- (2) For each pair of objects  $a_0, a_1 \in \text{Ob}\mathcal{A}$ , a graded  $\mathbb{K}$ -vector space  $\text{hom}(a_0, a_1)$ .
- (3) For  $k \geq 1$  and all sequences of  $k$  objects  $L_0, \dots, L_k$ , a map of degree  $2-k$

$$(3.4) \quad \mu^k: \text{hom}(L_{k-1}, L_k) \otimes \dots \otimes \text{hom}(L_0, L_1) \rightarrow \text{hom}(L_0, L_k)$$

satisfying the  $A_\infty$  **associativity relations**

$$(3.5) \quad \sum_{k=1}^d \sum_{i=1}^k \mu^k(\gamma_d, \dots, \gamma_{i+d-k+1}, \mu^{d-k+1}(\gamma_{i+d-k}, \dots, \gamma_i), \gamma_{i-1}, \dots, \gamma_1) = 0.$$

For an detailed treatment of  $A_\infty$ -categories, we refer the reader to chapter 1 of [33].

For  $\Sigma$  a boundary-punctured Riemann surface and  $\zeta \in \bar{\Sigma}$  a boundary puncture, a **positive strip-like end** is a holomorphic embedding

$$(3.6) \quad \epsilon: Z_+ = \mathbb{R}_{\geq 0} \times [0, 1] \rightarrow \Sigma$$

sending  $\mathbb{R}_{\geq 0} \times \{0\}$  and  $\mathbb{R}_{\geq 0} \times \{1\}$  to  $\partial\Sigma$ , and satisfying

$$\lim_{s \rightarrow \infty} \epsilon(s, t) = \zeta.$$

Similarly, a **negative strip-like end** for  $\zeta$  is a holomorphic embedding

$$(3.7) \quad \epsilon: Z_- = \mathbb{R}_{\leq 0} \times [0, 1] \rightarrow \Sigma$$

sending  $\mathbb{R}_{\leq 0} \times \{0\}$  and  $\mathbb{R}_{\leq 0} \times \{1\}$  to  $\partial\Sigma$ , and satisfying

$$\lim_{s \rightarrow -\infty} \epsilon(s, t) = \zeta.$$

If  $\Sigma^+$  has a positive strip-like end  $\epsilon^+$  and  $\Sigma^-$  has a negative strip-like end  $\epsilon^-$ , then we can glue  $\Sigma^+$  and  $\Sigma^-$  with length  $\ell > 0$  by removing  $\epsilon^+([\ell, \infty) \times [0, 1])$  and  $\epsilon^-((-\infty, -\ell] \times [0, 1])$  and identifying, for  $s \in (0, \ell)$ ,  $\epsilon^+(s, t)$  with  $\epsilon^-(s - \ell, t)$ . The resulting glued surface inherits any data on  $\Sigma^\pm$  supported away from the images of  $\epsilon^\pm$ .

A **boundary-punctured Riemann surface with strip-like ends** is a boundary-punctured Riemann surface  $\Sigma$ , along with a choice of a positive or negative strip-like end for each boundary



puncture, such that the images of the strip-like ends are pairwise disjoint. For a disk  $\Sigma^{d+1} \in \mathcal{R}^{d+1}$ , we require this to be a choice of strip-like end  $\epsilon_i$  for each  $\zeta_i$ , where  $\epsilon_i$  is positive for  $i > 0$  and negative for  $i = 0$ . Seidel has shown that we can make a **universal and consistent choice** of strip like ends: we can choose, for all  $d \geq 2$ , a collection of strip-like ends for each  $\Sigma^{d+1}$  varying smoothly over  $\mathcal{R}^{d+1}$ , and such that near  $\partial\overline{\mathcal{R}}^{d+1}$  they agree with the strip-like ends induced by gluing. See [33] for details.

A universal and consistent choice of strip-like ends gives rise to a **thick-thin decomposition** of each  $\Sigma^{d+1} \in \mathcal{R}^{d+1}$ , which we modify slightly from Seidel's convention. Namely, for a strip-like end  $\epsilon$ , define its  **$m$ -shift**  $\epsilon^m$  by

$$(3.8) \quad \epsilon^m(s, t) = \begin{cases} \epsilon(s + m, t) & \text{if } \epsilon \text{ is a positive strip-like end} \\ \epsilon(s - m, t) & \text{if } \epsilon \text{ is a negative strip-like end.} \end{cases}$$

Similarly, if  $S \in \Sigma^{d+1}$  is a finite-length strip obtained as the overlap from gluing  $\epsilon^+$  and  $\epsilon^-$  with length  $\ell$ , then  $S^m \subset S$  is the possibly empty finite-length strip obtained as the overlap from gluing  $(\epsilon^+)^m$  and  $(\epsilon^-)^m$  with length  $\ell - 2m$ . Now our thick-thin decomposition can be declared to be the 3-shift of Seidel's. In other words, the thin part of  $\Sigma^{d+1}$  is the union of the images of all 3-shifts of strip-like ends and all 3-shifts of gluing regions, and the thick part is its complement.

*Remark 3.7.* To be properly pedantic, one should first define a gluing region  $S$  of length  $\ell_S$  to be **good** if the strip-like ends induced by  $\Sigma^\pm$  agree with the strip-like ends on  $\Sigma$ , and to be **very good** if the corresponding gluing region is good and disjoint from all other good gluing regions for all lengths  $\ell \geq \ell_S$ . Then one defines the thin part to include only the 3-shifts of those gluing regions which are very good. This eliminates further choices from the setup and makes it easy to see that the thick part is nonempty.

In everything that follows, we will assume that that we've fixed a universal and consistent choice of strip-like ends.

Next, we recall Abouzaid's rescaling trick from [2]. Departing slightly from our earlier notation, let  $\phi^\tau$  be the diffeomorphism of  $\hat{M}$  given by the time  $\log \tau$  flow of the Liouville vector field. Note that pullback by  $\phi^\tau$  sends Lagrangians to Lagrangians, compatible Hamiltonians to compatible Hamiltonians, and preserves equation (3.1). Suppose then that we've fixed Lagrangians  $L_0$  and  $L_1$ , along with a nondegenerate Hamiltonian  $H$  and regular almost complex structure  $J$ . Then we get a natural bijection between solutions to (3.2) with boundary conditions  $(L_0, L_1)$  and solutions to

$$(3.9) \quad \partial_s u + (\phi^\tau)^* J(t) (\partial_t u - (\phi^\tau)^* X_H) = 0$$

with boundary conditions  $((\phi^\tau)^* L_0, (\phi^\tau)^* L_1)$ . The identity

$$(\phi^\tau)^* X_H = X_{\frac{1}{\tau}(\phi^\tau)^* H}$$

lets us rewrite (3.9) as

$$(3.10) \quad \partial_s u + J_\tau(t) (\partial_t u - X_{H_\tau}),$$

where

$$(J_\tau, H_\tau) = ((\phi^\tau)^* J, \frac{1}{\tau}(\phi^\tau)^* H)$$

again satisfies equation (3.1). Since  $H_\tau = \tau H$  near infinity, we can make  $H_\tau$  bigger than any other given compatible Hamiltonian by taking  $\tau$  sufficiently large.

Let us fix, for *each* pair of Lagrangians  $(L_i, L_j)$  in  $M$ , a nondegenerate Hamiltonian  $H^{i,j}$ , along with an almost complex structure  $J^{i,j} \in \mathcal{J}_{\text{reg}}^{[0,1]}(M, H^{i,j})$ . The pair  $(H^{i,j}, J^{i,j})$  is known as a Floer datum for  $(L_0, L_1)$ , and it singles out well defined wrapped and partially wrapped Floer complexes. In the sequel, this choice will be usually be implicit, and we will write, e.g.,  $\mathcal{X}(L_i, L_j)$  instead of  $\mathcal{X}(L_i, L_j, H^{i,j})$ .

For  $d \geq 2$  and a  $d + 1$ -tuple of Lagrangians  $(L_0, \dots, L_d)$ , we wish to define a family of maps

$$(3.11) \quad \mu^d: CW^*(L_{d-1}, L_d) \otimes \dots \otimes CW^*(L_0, L_1) \rightarrow CW^*(L_0, L_d)$$

of degree  $2 - d$  which satisfy an analog of Lemma 3.5. Let  $\Sigma \in \mathcal{R}^{d+1}$ . From our consistent and universal choice,  $\Sigma$  is equipped with a collection of strip-like ends. Let  $\partial_i \Sigma$  be the edge of  $\Sigma$  between  $\zeta_i$  and  $\zeta_{i+1}$ , or in the case  $i = d$  between  $\zeta_d$  and  $\zeta_0$ , and label  $\partial_i \Sigma$  with the Lagrangian  $L_i$ .

The following definition is important to the present situation, but we state it in enough generality that we won't need to rewrite it too many times.

**Definition 3.8.** A **Floer datum** on a boundary-punctured Riemann surface  $\Sigma$  with strip-like ends and Lagrangian labels consists of

- (1) A positive real number  $w_i$  for each puncture  $\zeta_i$ .
- (2) A sub-closed 1-form  $\beta$  on  $\Sigma$  satisfying  $\beta|_{\partial \Sigma} = 0$  and  $(\epsilon_i^1)^* \beta = w_i dt$  for all  $i$ .
- (3) A  $\Sigma$ -parametrized compatible Hamiltonian  $H$  on  $M$  satisfying a bunch of conditions:
  - (a)  $d^\Sigma H \wedge \beta \leq 0$  outside of a compact set. Here we view  $H$  as a function on  $\Sigma \times \hat{M}$ , and  $d^\Sigma H$  is the component of  $dH$  in the  $\Sigma$ -direction. Moreover,  $d^\Sigma H$  vanishes on outward normal vectors at  $\partial \Sigma$ , and  $d\beta$  is strictly negative and bounded away from zero on the support of  $d^\Sigma H$ .
  - (b) For each positive strip-like end  $\epsilon_i$ , let  $L_0$  and  $L_1$  be the Lagrangians assigned to the boundary components of  $\Sigma$  containing  $\epsilon_i(\mathbb{R}_{\geq 0} \times \{0\})$  and  $\epsilon_i(\mathbb{R}_{\geq 0} \times \{1\})$ , respectively. Then there is a scaling constant  $\tau_i > 0$  such that

$$w_i H = H_{\tau_i}^{0,1}$$

on the image of  $\epsilon_i$ .

- (c) For each negative strip-like end  $\epsilon_i$ , let  $L_0$  and  $L_1$  be the Lagrangians assigned to the boundary components of  $\Sigma$  containing  $\epsilon_i(\mathbb{R}_{\leq 0} \times \{0\})$  and  $\epsilon_i(\mathbb{R}_{\leq 0} \times \{1\})$ , respectively. Then there is a scaling constant  $\tau_i > 0$  such that

$$w_i H = H_{\tau_i}^{0,1}$$

on the image of  $\epsilon_i$ .

- (4) A  $\Sigma$ -parametrized almost complex structure  $J \in \mathcal{J}^\Sigma(M, H)$  such that
  - (a) For each strip-like end as above,  $J$  satisfies

$$J = J_{\tau_i}^{0,1}$$

on the image of  $\epsilon_i^2$ .

- (b) Let  $c: \Sigma \rightarrow \mathbb{R}_+$  be the constant in the compatibility condition (3.1). Then the support of  $dc$  is disjoint from the support of  $d^\Sigma H$ .
- (5) A smooth function  $\tau_E: \partial \Sigma \rightarrow (0, \infty)$  such that  $\tau_E(z) = \tau_i$  for all ends  $\zeta_i$  and all points  $z \in \partial \Sigma \cap \text{image}(\epsilon_i)$ .

For a family of Floer data, we strengthen condition 3a to require that  $d\beta$  is uniformly bounded away from zero on the support of  $d^\Sigma H$ .

A Floer datum for a boundary-punctured Riemann surface without Lagrangian labels consists of a Floer datum for every Lagrangian labeling of that Riemann surface.

**Lemma 3.9.** *Let  $\Sigma$  be a boundary-punctured Riemann surface with strip-like ends and Lagrangian labels. Then the space of Floer data on  $\Sigma$  is nonempty and contractible.*

*Proof.* For existence, choose  $\beta$ , which determines  $w_i$ . Then choose  $H$  to be  $\Sigma$ -independent outside of the strip-like ends. A choice of  $H$  determines  $\tau_i$ , and from there we can fill in choices of  $J$  and  $\tau_E$ .

For contractibility, we choose data in the order  $w_i$ , then  $\beta$ , then  $\tau_i$ , then  $H$ ,  $J$ , and  $\tau_E$ . Each space of choices forms a contractible set depending on the previous choices.  $\square$

Following Abouzaid, we consider conformal rescalings of Floer data. Namely, we say that the Floer data  $(\beta, H, J, \tau_E)$  and  $(\beta', H', J', \tau'_E)$  are conformally equivalent if there are constants  $C, W > 0$  such that

$$(3.12) \quad \beta = W\beta', \quad H = \frac{1}{W}(H')_C, \quad J = (J')_C, \quad \tau_E = C\tau'_E.$$

If  $\Sigma^+$  has a positive strip-like end  $\epsilon_i^+$  and  $\Sigma^-$  has a negative strip-like end  $\epsilon_j^-$ , and the corresponding Lagrangian labels agree, then Floer data on  $\Sigma^+$  and  $\Sigma^-$  can be glued by rescaling one and patching together the data. Specifically, one chooses  $C$  and  $W$  so that  $\tau_i^+ = C\tau_j^-$  and  $w_i^+ = Ww_j^-$  and uses those constants in (3.12) to define a new Floer datum on  $\Sigma^+$ . The precise Floer datum obtained by iterated gluing depends on the order of the gluings, but its conformal equivalence class does not.

We now specialize back to the disks with which we will construct the  $A_\infty$  structure. For that we will need coordinate charts near  $\partial\mathcal{R}^{d+1}$ , which we choose as in [33], but with the exponential gluing profile. In other words, if  $S \subset \partial\overline{\mathcal{R}}^{d+1}$  is a boundary stratum corresponding to a rooted tree  $T$  with labeled leaves, then a chart for  $\overline{\mathcal{R}}^{d+1}$  near  $\Sigma \in S$  is

$$(3.13) \quad \prod_{\text{internal vertices } v} U_v \times \prod_{\text{internal edges } e} [0, a_e).$$

Here,  $U_v$  is a subset of the space  $\mathcal{R}^{m+1}$  corresponding to the vertex  $v$ , and  $[0, a_e)$  is an interval of gluing parameters corresponding to the edge  $e$ , where gluing parameter  $\rho$  corresponds to the length  $\ell = e^{\frac{1}{\rho}}$ . The identity map from such a chart to one obtained from the logarithmic gluing profile  $\ell = \frac{-1}{\pi} \log \rho$  is smooth, and hence any smooth data on the classical associahedra can be pulled back to smooth data in these charts.

**Definition 3.10.** A **universal and conformally consistent** choice of Floer data for  $\mathcal{R}^{d+1}$  consists of, for all  $d \geq 2$ , a Floer datum  $\mathbf{K}(\Sigma) = (\beta, H, J, \tau_E)$  for each  $\Sigma$  varying smoothly over  $\mathcal{R}^{d+1}$ , and such that near  $\partial\overline{\mathcal{R}}^{d+1}$  it satisfies the following consistency condition.

- (1) For  $\Sigma$  sufficiently close to the boundary of  $\mathcal{R}^{d+1}$ ,  $\mathbf{K}(\Sigma)$  coincides on the thin part up to a conformal rescaling with the Floer datum induced by gluing.
- (2) In a chart of the form (3.13), we can consider the restriction of  $\mathbf{K}(\Sigma)$  to each piece  $\Sigma_i \in \mathcal{R}^{m+1}$  from which  $\Sigma$  is glued. This gives a family of Floer data on  $\Sigma_i$  parametrized by

$$U \times \prod_e (0, a_e) \times E,$$

where  $U \subset \mathcal{R}^{m+1}$  is a neighborhood of  $\Sigma_i$ , the intervals consist of the gluing parameters for gluing regions adjacent to  $\Sigma_i$ , and  $E$  contains all the remaining terms in (3.13). We require that this family extends smoothly to

$$U \times \prod_e [0, a_e) \times E,$$

and that on  $U \times \prod_e \{0\} \times E$  it agrees up to a family of conformal rescalings with the family of Floer data that was chosen for  $\mathcal{R}^{m+1}$ .

Though our situation is slightly different from Abouzaid's, Lemma 4.3 from [2] still holds, namely

**Lemma 3.11.** *Universal and conformally consistent choices of Floer data exist. Moreover, if  $\mathbf{K}_0$  is such a choice and  $K_\Sigma$  is another Floer datum on some  $\Sigma \in \mathcal{R}^{d+1}$ , then  $K_\Sigma$  can be extended to a universal and asymptotically consistent choice that agrees with  $\mathbf{K}_0$  on  $\mathcal{R}^{m+1}$  for all  $m < d$ .  $\square$*

Let  $(L_0, \dots, L_d)$  be a  $(d+1)$ -tuple of Lagrangians, and let

$$(3.14) \quad \gamma_i \in \begin{cases} \mathcal{X}(L_{i-1}, L_i) & i \neq 0 \\ \mathcal{X}(L_0, L_d) & i = 0. \end{cases}$$

Given a Floer datum  $\mathbf{K} = (\beta, H, J, \tau_E)$  on some  $\Sigma \in \mathcal{R}^{d+1}$ , we can consider maps  $u: \Sigma \rightarrow \hat{M}$  satisfying the generalized Floer equation

$$(3.15) \quad J \circ (du - X_H \otimes \beta) = (du - X_H \otimes \beta) \circ j$$

and such that  $u(\partial_i \Sigma) \subset (\phi^{\tau_E})^* L_i$  and  $u(\zeta_i) = (\phi^{\tau_i})^* \gamma_i$  in the sense of (3.3). More, given a universal and conformally consistent choice  $\mathbf{K}$ , we can consider  $\mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$ , the space of such maps as  $\Sigma$  varies in  $\mathcal{R}^{d+1}$  and  $K$  varies with  $\Sigma$ . Note that as with solutions to the Floer's equation on strips, a conformal rescaling of  $\mathbf{K}$  induces a *canonical* identification of the corresponding versions of  $\mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$ . In the sequel, we will usually make this identification implicitly.

Lemma A.8 shows that the maps  $u$  as above are constrained to take values in some compact part of  $\hat{M}$ , so that the Gromov compactness theorem applies. This says that  $\mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  has a natural compactification  $\overline{\mathcal{R}}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  obtained by adding in broken configurations similar to those for Floer trajectories. We enumerate those broken configurations with exactly two nonconstant components:

$$(3.16a) \quad \begin{aligned} & \mathcal{R}^{m+1+1}(\gamma_d, \dots, \gamma_{i+d-m+1}, \tilde{\gamma}, \gamma_i, \dots, \gamma_1; \gamma_0) & 1 \leq m \leq d-2 \\ & \times \mathcal{R}^{d-m+1}(\gamma_{i+d-m}, \dots, \gamma_{i+1}; \tilde{\gamma}) & 0 \leq i \leq m \\ & & \tilde{\gamma} \in \mathcal{X}(L_i, L_{i+d-m}) \end{aligned}$$

$$(3.16b) \quad \begin{aligned} & \mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_{i+1}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_1; \gamma_0) & 1 \leq i \leq d \\ & \times \mathcal{R}(\gamma_i; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}(L_{i-1}, L_i) \end{aligned}$$

$$(3.16c) \quad \mathcal{R}(\tilde{\gamma}; \gamma_0) \times \mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \tilde{\gamma}) \quad \tilde{\gamma} \in \mathcal{X}(L_0, L_d)$$

The first kind occur when a sequence of curves has domains approaching  $\partial \mathcal{R}^{d+1}$ , and the other two occur when energy escapes through one of the strip-like ends. The configurations with more than two components are in general some combination of the above, but since they don't show up in the construction of Fukaya categories, we won't worry about them. As with Floer trajectories in Lemma 3.4, there are only finitely many intermediate chords  $\tilde{\gamma}$  for which at least one of the above products is nonempty.

The key analytic ingredient is

**Lemma 3.12.** *There is a subset  $\mathcal{K}_{\text{reg}}(M)$  of the space of universal and conformally consistent choices of Floer data for  $\mathcal{R}^{d+1}$  which is dense and such that any  $\mathbf{K} \in \mathcal{K}_{\text{reg}}(M)$  has the following properties.*

- (1) *For all  $d \geq 2$ , all  $L_0, \dots, L_d$ , and all  $\gamma_i$  as in (3.14), the corresponding moduli space  $\mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  is a smooth manifold of dimension  $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) + d - 2$ .*
- (2) *If  $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = 2 - d$ , then  $\mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  is compact.*
- (3) *If  $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = 3 - d$ , then  $\overline{\mathcal{R}}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  is a compact topological 1-manifold with boundary, and its boundary is the union of all binary broken curves (3.16).*

*Proof.* The proof is explained in [33], Section 9.  $\square$

Fix an element  $\mathbf{K} \in \mathcal{K}_{\text{reg}}(M)$ , and hence a moduli space  $\mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  for all  $d$  and all  $\gamma_i$ . We can now define what will be the  $A_\infty$  operations  $\mu^d$ . Namely, one sets  $\mu^1$  to be the Floer

differential  $\delta$ , and

$$\mu^d(\gamma_d, \dots, \gamma_1) = \sum_{\substack{\gamma_0 \in \mathcal{X}(L_0, L_d) \\ \deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = 2-d}} \# \mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0) \cdot \gamma_0$$

if  $d \geq 2$ . One can check that, when they're zero-dimensional, the products in (3.16) encode all possible ways of composing two  $\mu^d$ 's with the given inputs and output. Since these make up the boundary of a compact 1-manifold, the total number of elements is even, so the  $\mu^k$  satisfy the  $A_\infty$  relations (3.5).

**Definition 3.13.** The **wrapped Fukaya category** of a pumpkin domain  $(M, \lambda_M, \sigma)$ , denoted  $\mathcal{W}(M, \lambda_M)$ , is the  $A_\infty$ -category whose objects are Lagrangians in  $M$ , in the sense of Definition 3.1, and such that  $\text{hom}(L_0, L_1) = CW^*(L_0, L_1)$ . The  $A_\infty$  structure is given by the  $\mu^d$  described above.

The **interior wrapped Fukaya category**  $\mathcal{W}^{int}(M, \lambda_M)$  is the full subcategory of the wrapped category containing only the interior Lagrangians of  $M$ .

*Remark 3.14.* Seidel has observed that while our wrapped Fukaya category of a pumpkin domain embeds as a full subcategory of the wrapped Fukaya category of the underlying Liouville domain, the latter category can have strictly more objects. These take the form of Lagrangian submanifolds which intersect the stops in an essential way. This is, however, impossible when  $M$  is a Weinstein domain.

As with the Floer differential, the disks defining the higher compositions have only isolated positive intersections with the divisors of the stops. The result is that the  $A_\infty$  operations preserve the intersection filtrations induced by the stops.

**Lemma 3.15.** *Let  $\sigma \in \sigma$  be a stop. Then, for any  $d \geq 1$  and composable sequence of morphisms  $\gamma_1, \dots, \gamma_d$ , we have*

$$n_\sigma(\mu^d(\gamma_d, \dots, \gamma_1)) \leq \sum_{i=1}^d n_\sigma(\gamma_i).$$

□

In particular, this says that the  $A_\infty$  operations preserve the partially wrapped complexes, so we can define

**Definition 3.16.** For  $(M, \lambda_M, \sigma)$  a pumpkin domain, its **partially wrapped Fukaya category**  $\mathcal{W}_\sigma(M, \lambda_M)$  is the subcategory of  $\mathcal{W}(M, \lambda_M)$  with all the same objects and such that

$$\text{hom}_{\mathcal{W}_\sigma}(L_0, L_1) = CW_\sigma^*(L_0, L_1)$$

The **interior partially wrapped Fukaya category**  $\mathcal{W}_\sigma^{int}(M, \lambda_M)$  is the full subcategory of the partially wrapped category containing only the interior Lagrangians of  $M$ .

As we will see, the two versions of the partially wrapped Fukaya category are in fact quasi-equivalent, but certain functors will be much easier to write down when we have access to both.

**3.3. Units and isomorphisms.** Here we give a mostly standard review of the construction of isomorphisms inside Fukaya categories. We do this because there are a number of places where it is important that these maps come from holomorphic curves. Specifically, we need to see that there are sufficiently strong maximum principles, as well as analogs of Lemmas 3.5 or 3.15.

The relevant situation is that we have a family of Lagrangians  $L_t$  parametrized by  $t \in [0, 1]$ . Such an isotopy can be generated by a Hamiltonian  $H_t$  which, for all  $t$ , is linear up to a term which is locally constant near the ends of  $L_t$ . This means there exists a linear Hamiltonian  $H_t^\ell$  whose Hamiltonian vector field agrees with that of  $H_t$  near the ends of  $L_t$ . We say that the family  $L_t$  is **transverse** if  $H_t^\ell$  can be taken to be transverse for all  $t$ .

To construct an isomorphism from  $L_0$  to  $L_1$ , we need to consider Riemann surfaces with moving Lagrangian labels. The starting point is Definition 3.8, which applies as long as the labels don't move inside the strip-like ends, but it isn't quite enough to achieve compactness. For that to work out, we need to control where the label moves and be a bit more careful about the sub-closed 1-form.

Additionally, for lack of a better maximum principle, we will construct holomorphic curves only for transverse families of Lagrangians. However, the proof of Lemma 2.17 shows that any family of Lagrangians can be  $C^0$ -approximated rel endpoints by a transverse family.

**Definition 3.17.** Let  $\Sigma$  be a boundary-punctured Riemann surface with strip-like ends. For us, a **moving Lagrangian label** on a component  $E$  of  $\partial\Sigma$  consists of a finite union of closed intervals  $T_E \subset E$  which avoids the images of the strip-like ends, along with a smooth transverse  $E$ -parametrized family of Lagrangians which is constant outside of  $T_E$ .

**Definition 3.18.** Let  $\Sigma$  be a boundary-punctured Riemann surface with strip-like ends and moving Lagrangian labels. A **Floer datum** for  $\Sigma$  consists of a 5-tuple  $(\beta, H, J, \tau_E, \beta^\ell)$  with the following properties:

- (1)  $(\beta, H, J, \tau_E)$  satisfies all of the conditions in Definition 3.8.
- (2) For each boundary component  $E \subset \partial\Sigma$ ,  $d\beta$  is strictly negative in a neighborhood of  $T_E$ .
- (3)  $\beta^\ell$  is a 1-form on  $\Sigma$  such that
  - (a) Outside of a fixed compact subset of  $\hat{M}$ ,  $d\beta$  is bounded away from zero on the support of  $\beta^\ell$ .
  - (b)  $\beta^\ell$  vanishes in the strip-like ends of  $\Sigma$ .
  - (c) Let  $c: \Sigma \rightarrow \mathbb{R}_+$  be the constant in the compatibility condition (3.1). Then the support of  $\beta^\ell$  is disjoint from the support of  $dc$ .
  - (d) There is a compact subset of  $\hat{M}$  outside of which, for any  $z \in \partial\Sigma$  and  $\xi \in T_z\partial\Sigma$ , the Hamiltonian vector field associated to  $\beta^\ell(\xi)\sqrt{H(z)}$  is tangent to the Lagrangian deformation associated to  $\xi$ .

For a family  $\mathcal{P}$  of Floer data, we require as usual that all compact subsets of  $\hat{M}$  appearing in this definition can be taken  $\mathcal{P}$ -independently.

If  $T_E$  is empty so that  $\Sigma$  has non-moving Lagrangian labels, then a Floer datum on  $\Sigma$  in the sense of Definition 3.8 induces one in the new sense by taking  $\beta^\ell = 0$ .

Before we get to specific instances of moving Lagrangians, it is worth considering the general features of holomorphic curves with moving Lagrangian boundary conditions. For such Floer data, the appropriate version of the holomorphic curve equation is

$$(3.17) \quad J \circ (du - X_H \otimes \beta - X_{\sqrt{H}} \otimes \beta^\ell) = (du - X_H \otimes \beta - X_{\sqrt{H}} \otimes \beta^\ell) \circ j$$

for a map  $u: \Sigma \rightarrow \hat{M}$  with boundary conditions given by the moving Lagrangian labels. Because we've restricted to transverse families of Lagrangians, all the expected properties survive. In particular, Lemma A.8 still holds, as does positivity of intersections:

**Lemma 3.19.** *Let  $\sigma \in \sigma$  be a stop. If  $\Sigma$  is connected, then any solution to (3.17) which is not contained in  $D_\sigma$  has only isolated positive intersections with  $D_\sigma$ .*  $\square$

Let  $D(L_t)$  be a disk with one negative boundary puncture  $\zeta$  and a moving Lagrangian label corresponding to the family  $L_t$ , such that in the coordinates of the strip-like end,  $\mathbb{R}_- \times \{0\}$  is labeled with  $L_0$  and  $\mathbb{R}_- \times \{1\}$  is labeled with  $L_1$ . Denote by  $\mathcal{K}^{D(L_t)}(M)$  the space of Floer data on  $D(L_t)$ . Then we can examine the regularity and compactness of the associated moduli spaces of holomorphic curves:

**Lemma 3.20.** *For  $\gamma \in \mathcal{X}(L_0, L_1)$ , let  $\mathcal{D}(L_t, \gamma)$  denote the space of solutions to (3.17) with boundary conditions given by the moving Lagrangian label, and which are asymptotic to  $(\phi^\tau)^*\gamma$  at  $\zeta$ . Then there is a comeager subset  $\mathcal{K}_{\text{reg}}^{D(L_t)}(M) \subset \mathcal{K}^{D(L_t)}(M)$  such that for any  $K \in \mathcal{K}_{\text{reg}}^{D(L_t)}(M)$ , the following hold.*

- (1) *For all  $\gamma$ ,  $\mathcal{D}(L_t, \gamma)$  is a smooth manifold of dimension  $\deg(\gamma)$ .*
- (2) *If  $\deg(\gamma) = 0$ , then  $\mathcal{D}(L_t, \gamma)$  is compact.*
- (3) *If  $\deg(\gamma) = 1$ , then  $\mathcal{D}(L_t, \gamma)$  has a Gromov compactification  $\overline{\mathcal{D}}(L_t, \gamma)$  which is a compact topological 1-manifold with boundary, and there is a canonical identification*

$$(3.18) \quad \partial \overline{\mathcal{D}}(L_t, \gamma) = \coprod_{\tilde{\gamma} \in \mathcal{X}(L_0, L_1)} \mathcal{R}(\tilde{\gamma}; \gamma) \times \mathcal{D}(\tilde{\gamma}).$$

*In this case,  $\tilde{\gamma}$  necessarily has degree 0.*

□

Fix a Floer datum  $K \in \mathcal{K}_{\text{reg}}^{D(L_t)}(M)$ , and hence a moduli space  $\mathcal{D}(L_t, \gamma)$  for all  $\gamma$ . Define an element  $e_{L_t} \in \text{hom}_{\mathcal{W}}(L_0, L_1)$  by

$$(3.19) \quad e_{L_t} = \sum_{\gamma \in \mathcal{X}(L_0, L_1)} \# \mathcal{D}(L_t, \gamma) \cdot \gamma.$$

By Lemma 3.19,  $\mathcal{D}(L_t, \gamma)$  is empty if  $n(\gamma) > 0$ , so that  $e_{L_t}$  in fact lies in  $\text{hom}_{\mathcal{W}_\sigma}(L_0, L_1)$ . Further, note that the right hand side of (3.18) precisely describes the coefficient of  $\gamma$  in  $\partial e_{L_t}$ , from which we conclude that  $e_{L_t}$  is closed. For  $L_t = L_0$  a constant family, we likewise obtain an element  $e_{L_0} \in \text{hom}_{\mathcal{W}_\sigma}(L_0, L_0)$ .

**Lemma 3.21.** *In the wrapped and partially wrapped Fukaya categories,  $e_{L_0}$  is a homology unit for  $L_0$ .*

*Proof.* We show only that  $\mu^2(|\gamma|, |e_{L_0}|) = |\gamma|$  for  $\gamma \in \text{hom}(L_0, L_1)$  closed, since the proof of the transposed identity is identical. To do this, we will use an interpolating family of holomorphic strips to construct a chain homotopy between  $\mu^2(e_{L_0}, \cdot)$  and  $\text{id}_{\text{hom}(L_0, L_1)}$ .

Let  $Z^{0,1}$  be the strip  $\mathbb{R} \times [0, 1]$  with  $\mathbb{R} \times \{0\}$  labeled by  $L_0$  and  $\mathbb{R} \times \{1\}$  labeled by  $L_1$ . Let  $K = (\beta_q, H_q, J_q, (\tau_E)_q, \beta_q^\ell)$  be a family of Floer data on  $Z^{0,1}$  parametrized by  $q \in (0, 1)$  and satisfying

- (1) As  $q$  approaches 0,  $K$  converges in  $C^\infty$  to  $(dt, H^{0,1}, J^{0,1}, 1, 0)$ .
- (2) For  $q$  sufficiently close to 1,  $K$  coincides up to conformal equivalence on the thin part, i.e. 3-shift of the gluing region, with the Floer datum induced by gluing  $\zeta \in D(L_0)$  to  $\zeta_1 \in \Sigma^{2+1}$  with gluing parameter  $1 - q$  (length  $\frac{-1}{\pi} \log(1 - q)$ ).
- (3) Near  $q = 1$ , the restriction of  $K$  to the pieces  $D(L_0)$  and  $\Sigma^{2+1}$  extends smoothly to a  $(1 - \epsilon, 1]$ -parametrized family which agrees up to conformal equivalence at  $q = 1$  with the previously chosen data.

As always, among the space of such Floer data, there is a comeager subset for which the moduli space  $\mathcal{Z}^{0,1}(\gamma; \gamma')$  of solutions to (3.17) with appropriate boundary and asymptotic conditions is a smooth manifold of dimension  $\deg(\gamma') - \deg(\gamma) + 1$ .

Fixing such a  $K$ , we consider the Gromov compactification of the above moduli space when it is 1-dimensional. This gives it the structure of a compact topological 1-manifold with boundary, and

the boundary consists of all 0-dimensional configurations of the following spaces:

$$\begin{aligned} & \tilde{\mathcal{R}}(\gamma; \gamma') \\ & \mathcal{R}^{2+1}(\gamma, \tilde{\gamma}; \gamma') \times \mathcal{D}(L_0, \tilde{\gamma}) \\ & \mathcal{R}(\tilde{\gamma}; \gamma') \times \mathcal{Z}^{0,1}(\gamma; \tilde{\gamma}) \\ & \mathcal{Z}^{0,1}(\tilde{\gamma}; \gamma') \times \mathcal{R}(\gamma; \tilde{\gamma}) \end{aligned}$$

The first two correspond to degenerations of the domain as  $q$  tends to 0 or 1, respectively, while the last two correspond to energy escaping out one of the strip-like ends. In algebraic terms, the first two correspond to  $\text{id}_{\text{hom}(L_0, L_1)}$  and  $\mu^2(e_{L_0}, \cdot)$ , respectively, while the last two describe a chain homotopy generated by  $\mathcal{Z}^{0,1}$ .  $\square$

**Corollary 3.22.** *The inclusion  $\mathcal{W}_{\sigma}(M) \rightarrow \mathcal{W}(M)$  is a cohomologically unital functor. In particular, any isomorphism in the partially wrapped Fukaya category is an isomorphism in the fully wrapped Fukaya category.*  $\square$

A similar argument, this time interpolating between  $\mu^2(e_{L_{1-t}}, e_{L_t})$  and  $e_{L_0}$ , shows

**Lemma 3.23.**  *$e_{L_t}$  is an isomorphism in the wrapped and partially wrapped Fukaya categories.*  $\square$

Using the fact that every Lagrangian is isotopic to an interior Lagrangian, and that any linear Hamiltonian isotopy can be approximated in  $C^0$  by a transverse isotopy, one obtains

**Corollary 3.24.** *The inclusion  $\mathcal{W}_{\sigma}^{\text{int}}(M, \lambda_M) \hookrightarrow \mathcal{W}_{\sigma}(M, \lambda_M)$  is a quasi-equivalence.*  $\square$

**3.4. Continuation functors.** Here, we sketch a construction of continuation maps and their enhancements to  $A_{\infty}$ -functors. These functors provide quasi-equivalences that relate the Fukaya categories obtained by making different universal and consistent choices of Floer data. Our construction aims to be efficient rather than elegant, and to this end we will build our moduli spaces starting with the boundary rather than the interior. The heuristic model which underlies all our choices is the space of ideal polygons in the hyperbolic upper half-plane with a corner at infinity, modulo translation. This space, suitably compactified, realizes Stasheff's multiplihedra. For other, nicer descriptions of the multiplihedra in the context of Floer theory, we refer the reader to [29].

To achieve associativity of strip-like end gluing in the boundary charts, we first need an auxiliary definition.

**Definition 3.25.** An **intrinsic width function** consists, for each  $d \geq 2$ , of  $d$  smooth functions  $w_i^d: \mathcal{R}^{d+1} \rightarrow \mathbb{R}_{\geq 0}$ ,  $i = 1, \dots, d$ , with the following properties:

- (1)  $w_1^2 = w_2^2 = 0$ .
- (2) Suppose  $\Sigma^{d+1}$  is obtained by gluing  $\zeta_n \in \Sigma^{k+1}$  to  $\zeta_0 \in \Sigma^{l+1}$  with length  $\ell$ . Then for all  $\ell$  sufficiently large, we have

$$w_i^d(\Sigma^{d+1}) = \begin{cases} w_i^k(\Sigma^{k+1}) & \text{if } i < n \\ w_{i+1-n}^l(\Sigma^{l+1}) + \ell & \text{if } n \leq i < n+l \\ w_{i+1-l}^k(\Sigma^{k+1}) & \text{if } i \geq n+l. \end{cases}$$

In other words, for  $\zeta_i$  not separated from  $\zeta_0$  by the long gluing region,  $w_i^d$  is unchanged, while for those that are,  $w_i^d$  increases by the length of the intervening gluing region.

Intrinsic width functions can be built by induction in  $d$ , and consistency near the corners of  $\overline{\mathcal{R}}^{d+1}$  amounts to the associativity of addition. Let us fix, once and for all, an intrinsic width function. We are now prepared to construct the multiplihedra as a compactified space of domains.



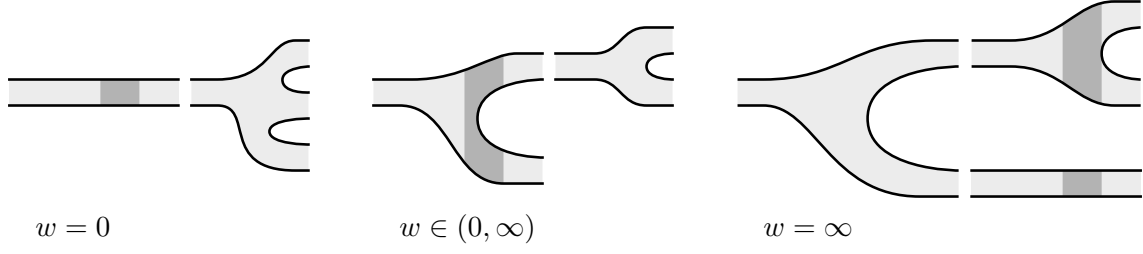


FIGURE 4. The types of boundary strata. The shaded regions correspond to where one thinks of the Floer data as changing from the data of the source to the data of the target. The disks with no shaded part are ordinary  $A_\infty$  disks.

*Construction 3.26.* Let  $\mathcal{S}^{1+1} = \{Z\}$ , where  $Z$  is equipped with a positive strip-like end  $\epsilon_+$  and a negative strip-like end  $\epsilon_-$  at  $+\infty$  and  $-\infty$ , respectively. For  $d \geq 2$  let  $\mathcal{S}^{d+1} = \mathcal{R}^{d+1} \times \mathbb{R}_+$ , where we temporarily forget all choices of strip-like ends. That is, we allow ourselves to have already chosen strip-like ends for a disk  $\Sigma \in \mathcal{R}^{d+1}$  but not for  $\Sigma$  thought of as an element  $(\Sigma, w) \in \mathcal{S}^{d+1}$ . Instead, we will construct a compactification  $\overline{\mathcal{S}}^{d+1}$  to be a model of the multiplihedron, and in doing so we will end up making a universal choice of strip-like ends  $\epsilon_{\mathcal{S}}$  for  $\mathcal{S}^{d+1}$ .

Before we begin, note that the multiplihedra are not in general manifolds with corners, but rather are a slightly more general type of smooth space. In addition to interior and boundary charts, it has charts parametrized by spaces of the form  $\mathbb{R}^m \times (V \cap [0, a)^n)$ , where  $m \leq d - 3$  and  $V \subset \mathbb{R}^n$  is a weighted homogeneous variety which is smooth on  $(0, a)^n$ . We will first describe the honest codimension 1 boundary, and then we will discuss how to fill in the generalized corners.

Suppose then by induction that we have constructed  $\overline{\mathcal{S}}^{d+1}$  to have boundary and generalized corners parametrized by associahedra and multiplihedra of lower dimension, and that we have consistently chosen strip-like ends on  $\Sigma$  for every  $(\Sigma, w) \in \mathcal{S}^{k+1}$  for  $k < d$ . Suppose further that these strip-like ends agree *up to shifts* with those chosen for  $\Sigma$  as an element of  $\mathcal{R}^{k+1}$ . We wish to do the same for  $\mathcal{S}^{d+1}$ .

Start by constructing a boundary chart which corresponds to taking  $w$  to 0, where  $w$  is the  $\mathbb{R}_+$  parameter. For this, take a small open subset  $U \subset \mathcal{R}^{d+1}$ . Then for a small and  $\rho \in (0, a)$ , we attach  $\mathcal{S}^{1+1} \times U \times [0, a)$  to  $\mathcal{S}^{d+1}$  via  $(Z, \Sigma, \rho) \mapsto (\Sigma, \rho)$ . The notation is intended to suggest gluing the positive end of  $Z$  to the negative end of  $\Sigma$ , and indeed doing this with length  $\ell = e^{\frac{1}{\rho}}$  induces a collection of strip-like ends on  $\Sigma$  thought of as  $(\Sigma, \rho) \in \mathcal{S}^{d+1}$ . Denote by  $\mathcal{S}_0^{d+1}$  the space obtained by adding to  $\mathcal{S}^{d+1}$  the above boundary faces.

Next, consider the boundary strata that appear when  $w$  tends to infinity. Here, a boundary chart is of the form  $U \times \prod_{i=1}^m U_i \times [0, a)$ , where  $U \subset \mathcal{R}^{m+1}$  and  $U_i \subset \mathcal{S}_0^{d_i+1}$ . To describe the attaching map for  $\rho \in (0, a)$ , we need to produce a pair  $(\Sigma, w)$  from the data  $(\Sigma^{m+1}, \{(\Sigma^{d_i+1}, w_i)\})$ . For the width, set  $w = e^{\frac{1}{\rho}}$ . For the surface, glue simultaneously and for all  $i$  the negative strip-like end of  $\Sigma^{d_i+1}$  to the positive strip-like end  $\epsilon_i$  in  $\Sigma^{m+1}$  with length  $\ell_i = w - w_i - w_i^m(\Sigma^{m+1})$ . This choice of gluing length ensures well defined corner charts when we compactify the  $\mathcal{R}^{m+1}$  and  $\mathcal{S}^{d_i+1}$  components. Once again, this gluing induces a choice of strip-like ends on  $\Sigma$  which varies with  $w$ .

The remaining boundary charts are obtained by compactifying all of the  $\mathcal{R}^{k+1}$  components. These appear either at  $w = 0$  or  $\infty$  on their own, or at  $w \in (0, \infty]$  as the non-width part of some  $\mathcal{S}^{k+1}$ . For an  $\mathcal{R}^{k+1}$  coming on its own, compactify it as the associahedron that it is. For the others, following the notation of (3.13), a boundary chart for  $\mathcal{S}^{k+1}$  is of the form

$$U_{\text{root}} \times \prod_{\substack{\text{non-root} \\ \text{internal vertices } v}} U_v \times \prod_{\text{internal edges } e} [0, a_e).$$

Here,  $U_v$  is as in (3.13), while,  $U_{root}$  is a small open subset of  $\mathcal{S}^{m+1}$ , where  $m$  is the valency of the root vertex. When all gluing parameters are nonzero, the identification with a subset of  $\mathcal{S}^{d+1}$  is as for  $\overline{\mathcal{R}}^{d+1}$ , with the extra  $\mathbb{R}_+$ -factor in  $\mathcal{S}^{m+1}$  mapping to the corresponding factor in  $\mathcal{S}^{d+1}$  via identity. This gluing induces strip-like ends on  $\Sigma$  for  $(\Sigma, w)$  near the remaining boundary components, and it remains only to attach the generalized corners and choose strip-like ends on the interior.

The compactification at  $w \in (0, \infty)$  also gives rise to ordinary corners, for which the inductive hypothesis guarantees that the induced strip-like ends don't depend on the order of gluing. Similarly, we compactify the  $\mathcal{R}^{d+1}$  component at  $w = 0$  and obtain more ordinary corners, and again the inductive hypothesis guarantees consistency. It remains to consider the corners at  $w = +\infty$ . When we looked at the codimension 1 portion of this limit, we enforced a correlation among gluing lengths which allowed us to associate a number  $w$  to the glued disk. For the corners, we instead consider the space  $(0, a)^n$  of all possible combinations of gluing parameters, and observe that the correlations give rise to a submanifold of  $(0, a)^n$  for which  $w$  is well-defined. This submanifold naturally closes to a singular submanifold of  $[0, a)^n$  which is topologically a manifold with boundary. Along with terms associated to the interiors of  $\mathcal{R}^{m+1}$  and  $\mathcal{S}^{k+1}$ , this provides the desired chart for a generalized corner.

To complete the inductive construction, simply choose a collection of  $\mathcal{S}^{d+1}$ -parametrized shifts which, when applied to the strip-like ends on  $\mathcal{R}^{d+1}$ , interpolate between those we constructed near  $\partial\mathcal{S}^{d+1}$ .

*Remark 3.27.* While the boundary charts for  $\mathcal{S}^{d+1}$  depended on a family of choices, the smooth structure does not. To see this, note first that any two choices differ by a collection of smooth families of shifts. In the boundary charts, these can be corrected by modifying the gluing parameters and shifting the implicit width function.

Suppose we had, for each pair of Lagrangians  $(L_i, L_j)$ , two choices of Floer data  $(H_0^{i,j}, J_0^{i,j})$  and  $(H_1^{i,j}, J_1^{i,j})$ . Let  $\mathbf{K}_0$  and  $\mathbf{K}_1$  be universal and conformally consistent choices of Floer data for  $\mathcal{R}^{d+1}$  which are built from these data. For the choice  $(\mathbf{K}_\nu, \mathbf{J}_\nu)$ , we will denote the resulting Floer-theoretic doodads with a subscript of  $\nu$ , e.g.  $\mathcal{X}_\nu(L_i, L_j)$ ,  $\mathcal{R}_\nu(\gamma_+; \gamma_-)$ ,  $CW_\nu^*(L_i, L_j)$ , or  $\mathcal{W}_{\sigma, \nu}(M, \lambda_M)$ .

Fixing a universal choice of strip-like ends  $\epsilon_S$ , we proceed to choose data on  $\mathcal{S}^{d+1}$  which interpolates between  $\mathbf{K}_0$  and  $\mathbf{K}_1$ . Namely, suppose we are given a  $(d+1)$ -tuple of Lagrangians  $(L_0, \dots, L_d)$ , which induces Lagrangian labels on every disk component of every point of  $\overline{\mathcal{S}}^{d+1}$ . In this situation, a Floer datum on  $\Sigma$  consists of a 4-tuple  $(\beta, H, J, \tau_E)$  which satisfies Definition 3.8, except that in conditions 3b and 4a for the positive ends, we replace  $(H^{0,1}, J^{0,1})$  with  $(H_0^{0,1}, J_0^{0,1})$ , while in conditions 3c and 4a for the negative ends, we replace  $(H^{0,1}, J^{0,1})$  with  $(H_1^{0,1}, J_1^{0,1})$ . In other words, we use the “0” data for the inputs and the “1” data for the output.

Consider the space  $\mathcal{K}^S(M)$  of **universal and conformally consistent** choices of Floer data for  $\mathcal{S}^{d+1}$ . Elements  $\mathbf{K} \in \mathcal{K}^S(M)$  consist of a choice, for each  $(\Sigma, w) \in \mathcal{S}^{d+1}$ , of a Floer datum  $\mathbf{K}(\Sigma, w)$  on  $\Sigma$  such that the family varies smoothly on  $\mathcal{S}^{d+1}$  and satisfies the following analog of Definition 3.10:

(3.20) Near each boundary stratum as parametrized in Construction 3.26,  $\mathbf{K}(\Sigma, w)$  coincides up to conformal equivalence on the  $\epsilon_S$ -thin part with the Floer datum determined by gluing, where all Floer data for disks  $\Sigma^{m+1} \in \mathcal{R}^{m+1}$  belong to whichever of  $\mathbf{K}_0$  or  $\mathbf{K}_1$  will make them a priori gluable. On the thick part, the restriction of  $\mathbf{K}(\Sigma, w)$  to each piece extends smoothly to the boundary, where it is conformally equivalent to the previously chosen Floer datum.

For such a choice  $\mathbf{K}$ , one can consider

$$(3.21) \quad \gamma_i \in \begin{cases} \mathcal{X}_0(L_{i-1}, L_i) & i \neq 0 \\ \mathcal{X}_1(L_0, L_d) & i = 0. \end{cases}$$

and the corresponding moduli space  $\mathcal{S}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  of solutions to (3.15) with the obvious boundary and asymptotic conditions. This space has a Gromov compactification  $\overline{\mathcal{S}}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  obtained by adding in appropriate broken configurations. We enumerate those broken configurations with only one  $\mathcal{R}$  or  $\mathcal{R}^{d+1}$  component.

$$(3.22a) \quad \begin{aligned} & \mathcal{R}_1^{k+1}(\tilde{\gamma}_k, \dots, \tilde{\gamma}_1; \gamma_0) & m_i \geq 1, \sum m_i = d \\ & \times \prod_{i=1}^k \mathcal{S}^{m_i+1}(\gamma_{\sum_{j=1}^i m_j}, \dots, \gamma_{\sum_{j=1}^{i-1} m_j+1}; \tilde{\gamma}_i) & \tilde{\gamma}_i \in \mathcal{X}_1(L_{\sum_{j=1}^{i-1} m_j}, L_{\sum_{j=1}^i m_j}) \end{aligned}$$

$$(3.22b) \quad \begin{aligned} & \mathcal{S}^{m+1+1}(\gamma_d, \dots, \gamma_{i+d-m+1}, \tilde{\gamma}, \gamma_i, \dots, \gamma_1; \gamma_0) & 0 \leq i \leq m \leq d-2 \\ & \times \mathcal{R}_0^{d-m+1}(\gamma_{i+d-m}, \dots, \gamma_{i+1}; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}_0(L_i, L_{i+d-m}) \end{aligned}$$

$$(3.22c) \quad \begin{aligned} & \mathcal{S}^{d+1}(\gamma_d, \dots, \gamma_{i+1}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_1; \gamma_0) & 1 \leq i \leq d \\ & \times \mathcal{R}_0(\gamma_i; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}_0(L_{i-1}, L_i) \end{aligned}$$

$$(3.22d) \quad \mathcal{R}_1(\tilde{\gamma}; \gamma_0) \times \mathcal{S}^{d+1}(\gamma_d, \dots, \gamma_1; \tilde{\gamma}) \quad \tilde{\gamma} \in \mathcal{X}_1(L_0, L_d)$$

The first type comes from the boundary component of  $\overline{\mathcal{S}}^{d+1}$  where  $w$  tends to infinity, while the second comes from  $w$  finite or zero. The other two come from energy escaping through the strip-like ends. The same transversality argument as in Lemma 3.12 gives

**Lemma 3.28.** *There is a subset  $\mathcal{K}_{reg}^{\mathcal{S}}(M) \subset \mathcal{K}^{\mathcal{S}}(M)$  which is dense and such that any  $\mathbf{K} \in \mathcal{K}_{reg}^{\mathcal{S}}(M)$  has the following properties.*

- (1) For all  $d \geq 1$ , all  $L_0, \dots, L_d$ , and all  $\gamma_i$  as in (3.21), the corresponding moduli space  $\mathcal{S}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  is a smooth manifold of dimension  $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) + d - 1$ .
- (2) If  $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = 1 - d$ , then  $\mathcal{S}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  is compact.
- (3) If  $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = 2 - d$ , then  $\overline{\mathcal{S}}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  is a compact topological 1-manifold with boundary, and its boundary is the union of all broken curves which appear in (3.22).

*Remark on proof.* The proof of the first two parts is standard. For the third part, when the broken configurations involve only one gluing region, the proof is identical to the proof for the  $A_\infty$  operations. For configurations involving several gluing regions with correlated lengths, we proceed as follows. To start, note that an index count shows that the configurations in (3.22a) are the only ones which can occur. Thus, we consider the full corner  $[0, a)^k$  in which the space of allowed gluing parameters is a 1-dimensional subvariety. By the Whitney extension theorem, we may extend our Floer data to this larger space of domains, and by rerunning the transversality argument we may assume the extension is regular. Since there is a broken curve for which each component occurs in index 0, the analytic gluing map with domain  $(0, a')^k$  for  $a' \ll a$  is bijective to the index  $k$  portion of the larger moduli space of maps with the appropriate subset of domains (not just those obtained by gluing with the given lengths, but also by perturbing the unglued domains in  $\mathcal{R}^{k+1}$  and  $\mathcal{S}^{m_i+1}$ ). Taking the 1-dimensional subset with correlated gluing lengths then gives the corresponding end of  $\mathcal{S}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$ .  $\square$

For  $d \geq 1$ , define  $\mathcal{F}^d: CW_0^*(L_{d-1}, L_d) \otimes \cdots \otimes CW_0^*(L_0, L_1) \rightarrow CW_1^*(L_0, L_d)$  by

$$(2.23) \quad \mathcal{F}^d(\gamma_d, \dots, \gamma_1) = \sum_{\substack{\gamma_0 \in \mathcal{X}_1(L_0, L_1) \\ \deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = 1-d}} \# \mathcal{S}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0) \cdot \gamma_0.$$

As usual, the characterization in Lemma 3.28 of the boundary of the 1-dimensional components of  $\mathcal{S}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  implies that

$$\begin{aligned} & \sum_{m=0}^{d-1} \sum_{i=0}^m \mathcal{F}^{m+1} \left( \gamma_d, \dots, \gamma_{i+d-m+1}, \mu_0^{d-m}(\gamma_{i+d-m}, \dots, \gamma_{i+1}), \gamma_i, \dots, \gamma_1 \right) \\ & + \sum_{k=1}^d \sum_{\substack{m_i \geq 1 \\ \sum_{i=1}^k m_i = d}} \mu_1^k \left( \mathcal{F}^{m_k}(\gamma_{\sum_{j=1}^k m_j}, \dots, \gamma_{\sum_{j=1}^{k-1} m_j + 1}), \dots, \mathcal{F}^{m_1}(\gamma_{m_1}, \dots, \gamma_1) \right) = 0 \end{aligned}$$

This collection of equations, as  $d$  ranges over the positive integers, is precisely the condition that  $\mathcal{F}$  is an  $A_\infty$ -functor from  $\mathcal{W}_0(M, \lambda_M)$  to  $\mathcal{W}_1(M, \lambda_M)$ .

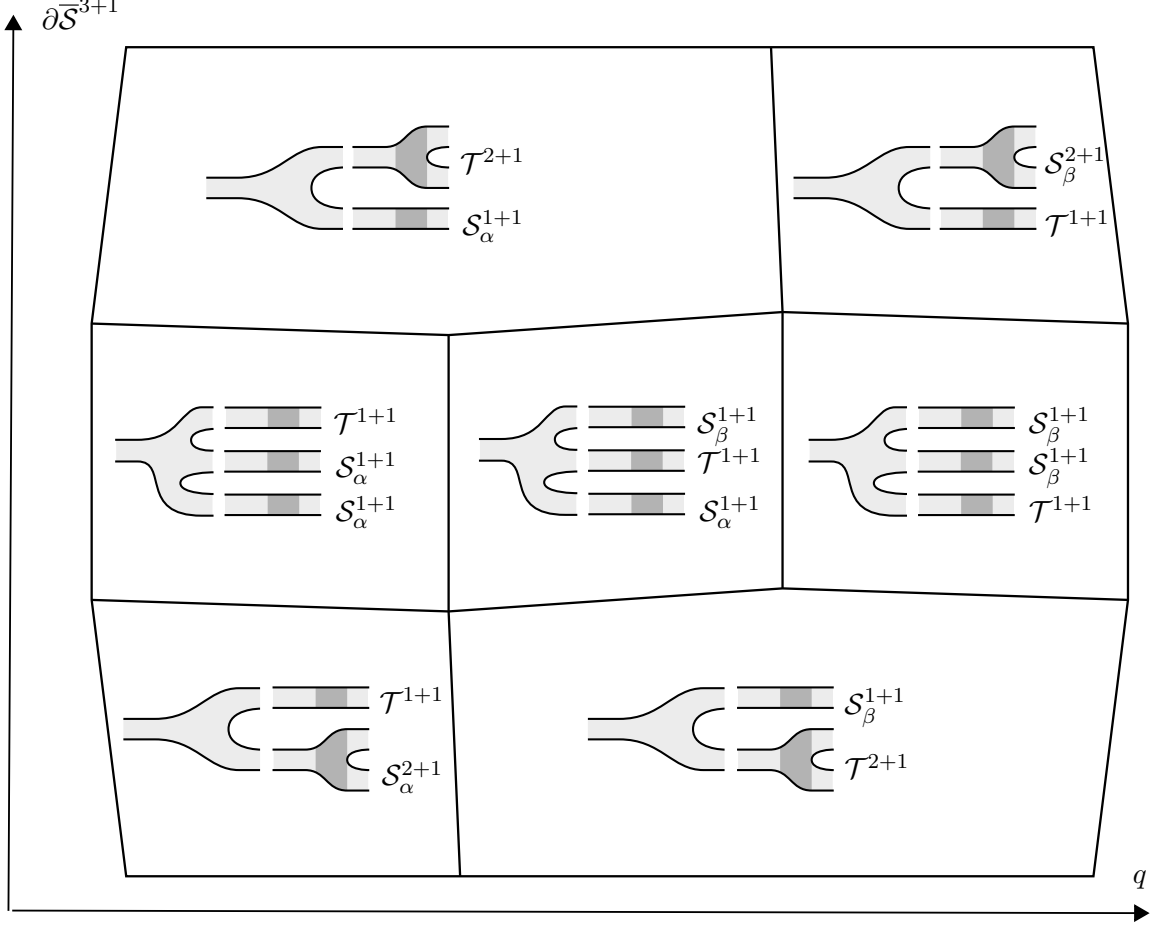
**Definition 3.29.**  $\mathcal{F}$  is called the **continuation functor** determined by  $(\mathbf{K}, \mathbf{J})$ .

By positivity of intersections, we see as in Lemma 3.5 that  $\mathcal{F}^d$  sends  $\mathcal{W}_{\sigma,0}(M, \lambda_M)$  to  $\mathcal{W}_{\sigma,1}(M, \lambda_M)$ , so that  $\mathcal{F}$  restricts to a functor on partially wrapped Fukaya categories. We also call this restricted functor  $\mathcal{F}$ , and still refer to it as a continuation functor.

**3.5. Homotopies between continuation functors.** To show that continuation functors are quasi-equivalences, it is enough to construct homotopies on the linear part, since that shows that they induce isomorphisms on homology. However, to prove that partially wrapped Fukaya categories are invariant under isotopies of the stops, it will be convenient to construct a full  $A_\infty$  homotopy between a given pair of continuation functors. To this end, we sketch the construction of a sequence of spaces which parametrize homotopies in the same way that the associahedra and multiplihedra parametrize composition and maps. We call these spaces **homotohedra** for lack of a better term, though we won't prove that they're polyhedra.

*Construction 3.30.* The construction of the  $d$ th homotohedron  $\overline{\mathcal{T}}^{d+1}$  is similar to that for the multiplihedron in Construction 3.26. Namely, one takes the space  $\mathcal{S}^{d+1}$ , adds in an extra parameter, and compactifies with boundary strata that manifestly induce the  $A_\infty$ -homotopy equations. In this case, things are even easier than before, since as smooth spaces we can take  $\mathcal{T}^{d+1} = \mathcal{S}^{d+1} \times (0, d)$  and  $\overline{\mathcal{T}}^{d+1} = \overline{\mathcal{S}}^{d+1} \times [0, d]$ . The trickiness is in choosing the strip-like ends.

To begin, suppose we are given two universal choices of strip-like ends  $\epsilon_S^\alpha$  and  $\epsilon_S^\beta$  for the multiplihedra, which we think of as two different models of the multiplihedron  $\mathcal{S}_\alpha^{d+1}$  and  $\mathcal{S}_\beta^{d+1}$ . For  $d = 1$ , pick a family of strip-like ends on  $\mathcal{Z}$  parametrized by  $(Z, q) \in \mathcal{T}^{1+1}$  which coincides with  $\epsilon_S^\alpha$  near  $q = 0$  and with  $\epsilon_S^\beta$  near  $q = 1$ . Thus, we identify  $(Z, 0) \in \overline{\mathcal{T}}^{1+1}$  with  $Z \in \mathcal{S}_\alpha^{1+1}$  and  $(Z, 1) \in \overline{\mathcal{T}}^{1+1}$  with  $Z \in \mathcal{S}_\beta^{1+1}$ . For  $d > 1$ , we likewise identify  $\overline{\mathcal{S}}^{d+1} \times \{0\}$  with  $\overline{\mathcal{S}}_\alpha^{d+1}$  and  $\overline{\mathcal{S}}^{d+1} \times \{d\}$  with  $\overline{\mathcal{S}}_\beta^{d+1}$ , and we choose strip-like ends in a small neighborhood of these faces to be independent of the  $[0, d]$ -parameter. We aim to extend this choice to  $\partial \overline{\mathcal{S}}^{d+1} \times (0, d)$  as follows. The boundary at  $w = 0$  is of the form  $\mathcal{S}^{1+1} \times \mathcal{R}^{d+1} \times (0, d)$ , which we identify with  $\mathcal{T}^{1+1} \times \mathcal{R}^{d+1}$  via  $(Z, \Sigma, q) \mapsto ((Z, \frac{q}{d}), \Sigma)$ . The gluing chart is as with the associahedron and doesn't change  $q$ . The boundary at  $w = +\infty$  is more complicated. Here, unlike with the associahedra, the boundary face is identified with a product of associahedra, multiplihedra, and homotohedra which change *discretely* with  $q$ . Specifically, suppose

FIGURE 5. The  $w = +\infty$  boundary of  $\bar{\mathcal{T}}^{3+1}$ .

we are in a face of  $\bar{\mathcal{T}}$  of the form

$$U \times \prod_{i=1}^m U_i \times (0, d)$$

where  $U \subset \mathcal{R}^{m+1}$  and  $U_i \subset \mathcal{S}^{d_i+1}$ . Then, for  $q \in \left(\sum_{i=j+1}^m d_i, \sum_{i=j}^m d_i\right)$ , we make the identification

$$(3.24) \quad \mathcal{S}^{d_i+1} = \begin{cases} \mathcal{S}_\alpha^{d_i+1} & \text{if } i < j \\ \mathcal{S}_\alpha^{d_j+1} \times \left\{q - \sum_{i=j+1}^m d_i\right\} \subset \mathcal{T}^{d_j+1} & \text{if } i = j \\ \mathcal{S}_\beta^{d_i+1} & \text{if } i > j. \end{cases}$$

For  $q$  of the form  $\sum_{i=j+1}^m d_i$  for some  $j$ , we make the simpler identification

$$(3.25) \quad \mathcal{S}^{d_i+1} = \begin{cases} \mathcal{S}_\alpha^{d_i+1} & \text{if } i \leq j \\ \mathcal{S}_\beta^{d_i+1} & \text{if } i > j. \end{cases}$$

See Figure 5.

If, for  $k < d$ , we have chosen strip-like ends for  $\mathcal{T}^{k+1}$  which agree near the above boundary faces with those given by gluing, we can do the same for  $\mathcal{T}^{d+1}$  by induction. Extending this family of strip-like ends arbitrarily to the interior, we obtain the desired choice.

Let  $\mathbf{K}_\alpha$  and  $\mathbf{K}_\beta$  be universal and conformally consistent choices of Floer data for  $\mathcal{S}_\alpha^{d+1}$  and  $\mathcal{S}_\beta^{d+1}$ , respectively. Assume that they both interpolate between  $\mathbf{K}_0$  and  $\mathbf{K}_1$ , and that they are both regular in the sense of Lemma 3.28. As with continuation functors, consider the space  $\mathcal{K}^\mathcal{T}(M)$  whose elements  $\mathbf{K}_\mathcal{T}$  are families of Floer data for  $\mathcal{T}^{d+1}$  which are universal and conformally consistent in the obvious way, with the following strengthening:

(3.26) In Definition 3.10, we ask that the smooth extension to the  $q = 0$  or  $d$  boundary strata agrees to infinite order to the family of Floer data induced by gluing.

This ensures that any consistent choice for  $\mathcal{T}^{k+1}$  for  $k < d$  can be extended to  $\mathcal{T}^{d+1}$ . Specifically, it avoids the danger of non-smoothness near the interface strata (3.25). Letting  $\mathbf{K}_\mathcal{T}$  denote such a choice, we examine the corresponding spaces of holomorphic curves  $\mathcal{T}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$ . Here,  $\gamma_i$  satisfy (3.21) as for functors. This space has a Gromov compactification  $\overline{\mathcal{T}}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  obtained by adding in possibly-broken configurations. As usual, these include terms which can come from either energy escape through the ends or domain degeneration to the boundary of  $\mathcal{T}^{d+1}$ . In this case, however, some of the new objects include ordinary non-broken disks coming from the  $q = 0$  and  $q = d$  components. We enumerate those new configurations with at most one  $\mathcal{R}$  or  $\mathcal{R}^{d+1}$  component.

$$\begin{aligned}
 & \mathcal{R}_1^{k+1}(\tilde{\gamma}_k, \dots, \tilde{\gamma}_1; \gamma_0) \\
 & \times \prod_{i=1}^{r-1} \mathcal{S}_\alpha^{m_i+1} \left( \gamma_{\sum_{j=1}^i m_j}, \dots, \gamma_{\sum_{j=1}^{i-1} m_j+1}; \tilde{\gamma}_i \right) \quad \begin{array}{l} 1 \leq r \leq k \leq d \\ m_i \geq 1, \sum m_i = d \end{array} \\
 & \times \mathcal{T}^{m_r+1} \left( \gamma_{\sum_{j=1}^r m_j}, \dots, \gamma_{\sum_{j=1}^{r-1} m_j+1}; \tilde{\gamma}_r \right) \quad \tilde{\gamma}_i \in \mathcal{X}_1 \left( L_{\sum_{j=1}^{i-1} m_j}, L_{\sum_{j=1}^i m_j} \right) \\
 & \times \prod_{i=r+1}^k \mathcal{S}_\beta^{m_i+1} \left( \gamma_{\sum_{j=1}^i m_j}, \dots, \gamma_{\sum_{j=1}^{i-1} m_j+1}; \tilde{\gamma}_i \right)
 \end{aligned} \tag{3.27a}$$

$$\begin{aligned}
 & \mathcal{T}^{m+1+1}(\gamma_d, \dots, \gamma_{i+d-m+1}, \tilde{\gamma}, \gamma_i, \dots, \gamma_1; \gamma_0) \quad 0 \leq i \leq m \leq d-2 \\
 & \times \mathcal{R}_0^{d-m+1}(\gamma_{i+d-m}, \dots, \gamma_{i+1}; \tilde{\gamma}) \quad \tilde{\gamma} \in \mathcal{X}_0(L_i, L_{i+d-m})
 \end{aligned} \tag{3.27b}$$

$$\begin{aligned}
 & \mathcal{T}^{d+1}(\gamma_d, \dots, \gamma_{i+1}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_1; \gamma_0) \quad 1 \leq i \leq d \\
 & \times \mathcal{R}_0(\gamma_i; \tilde{\gamma}) \quad \tilde{\gamma} \in \mathcal{X}_0(L_{i-1}, L_i)
 \end{aligned} \tag{3.27c}$$

$$\mathcal{R}_1(\tilde{\gamma}; \gamma_0) \times \mathcal{T}^{d+1}(\gamma_d, \dots, \gamma_1; \tilde{\gamma}) \quad \tilde{\gamma} \in \mathcal{X}_1(L_0, L_d) \tag{3.27d}$$

$$\mathcal{S}_\alpha^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0) \tag{3.27e}$$

$$\mathcal{S}_\beta^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0) \tag{3.27f}$$

$$\begin{aligned}
 & \mathcal{R}_1^{k+1}(\tilde{\gamma}_k, \dots, \tilde{\gamma}_1; \gamma_0) \\
 & \times \prod_{i=1}^r \mathcal{S}_\alpha^{m_i+1} \left( \gamma_{\sum_{j=1}^i m_j}, \dots, \gamma_{\sum_{j=1}^{i-1} m_j+1}; \tilde{\gamma}_i \right) \quad \begin{array}{l} 0 \leq r \leq k \\ 1 \leq k \leq d \\ m_i \geq 1, \sum m_i = d \end{array} \\
 & \times \prod_{i=r+1}^k \mathcal{S}_\beta^{m_i+1} \left( \gamma_{\sum_{j=1}^i m_j}, \dots, \gamma_{\sum_{j=1}^{i-1} m_j+1}; \tilde{\gamma}_i \right) \quad \tilde{\gamma}_i \in \mathcal{X}_1 \left( L_{\sum_{j=1}^{i-1} m_j}, L_{\sum_{j=1}^i m_j} \right)
 \end{aligned} \tag{3.27g}$$

$$(3.27h) \quad \begin{aligned} & \mathcal{S}_\alpha^{m+1+1}(\gamma_d, \dots, \gamma_{i+d-m+1}, \tilde{\gamma}, \gamma_i, \dots, \gamma_1; \gamma_0) & 0 \leq i \leq m \leq d-2 \\ & \times \mathcal{R}_0^{d-m+1}(\gamma_{i+d-m}, \dots, \gamma_{i+1}; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}_0(L_i, L_{i+d-m}) \end{aligned}$$

$$(3.27i) \quad \begin{aligned} & \mathcal{S}_\beta^{m+1+1}(\gamma_d, \dots, \gamma_{i+d-m+1}, \tilde{\gamma}, \gamma_i, \dots, \gamma_1; \gamma_0) & 0 \leq i \leq m \leq d-2 \\ & \times \mathcal{R}_0^{d-m+1}(\gamma_{i+d-m}, \dots, \gamma_{i+1}; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}_0(L_i, L_{i+d-m}) \end{aligned}$$

$$(3.27j) \quad \begin{aligned} & \mathcal{S}_\alpha^{d+1}(\gamma_d, \dots, \gamma_{i+1}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_1; \gamma_0) & 1 \leq i \leq d \\ & \times \mathcal{R}_0(\gamma_i; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}_0(L_{i-1}, L_i) \end{aligned}$$

$$(3.27k) \quad \begin{aligned} & \mathcal{S}_\beta^{d+1}(\gamma_d, \dots, \gamma_{i+1}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_1; \gamma_0) & 1 \leq i \leq d \\ & \times \mathcal{R}_0(\gamma_i; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}_0(L_{i-1}, L_i) \end{aligned}$$

$$(3.27l) \quad \mathcal{R}_1(\tilde{\gamma}; \gamma_0) \times \mathcal{S}_\alpha^{d+1}(\gamma_d, \dots, \gamma_1; \tilde{\gamma}) \quad \tilde{\gamma} \in \mathcal{X}_1(L_0, L_d)$$

$$(3.27m) \quad \mathcal{R}_1(\tilde{\gamma}; \gamma_0) \times \mathcal{S}_\beta^{d+1}(\gamma_d, \dots, \gamma_1; \tilde{\gamma}) \quad \tilde{\gamma} \in \mathcal{X}_1(L_0, L_d)$$

Here, one should think of (3.27a)-(3.27d) as completely analogous to (3.22). The next two, (3.27e) and (3.27f), correspond to degenerations  $q \rightarrow 0$  and  $q \rightarrow d$ . (3.27g) comes from taking  $w \rightarrow \infty$  as in (3.22a), except that  $q$  takes a nongeneric value. The remaining configurations come from taking  $q$  to 0 or  $d$  in (3.27b)-(3.27d). The usual transversality argument gives

**Lemma 3.31.** *There is a subset  $\mathcal{K}_{reg}^\mathcal{T}(M) \subset \mathcal{K}^\mathcal{T}(M)$  which is dense and such that any  $\mathbf{K} \in \mathcal{K}_{reg}^\mathcal{T}(M)$  has the following properties.*

- (1) For all  $d \geq 1$ , all  $L_0, \dots, L_d$ , and all  $\gamma_i$  as in (3.21), the corresponding moduli space  $\mathcal{T}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  is a smooth manifold of dimension  $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) + d$ .
- (2) If  $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = -d$ , then  $\mathcal{T}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  is compact.
- (3) If  $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = 1 - d$ , then  $\overline{\mathcal{T}}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  is a compact topological 1-manifold with boundary, and its boundary is the union of all configurations of the form (3.27a)-(3.27f).

*Remark on proof.* For the most part, the proof is identical to that for the moduli spaces associated to functors in Lemma 3.28. In this case, there are also many broken configurations with only one gluing parameter which don't appear. This happens because they correspond to isolated values of  $q$ , and varying  $q$  gives another deformation parameter. Therefore, they occur in codimension at least two. Alternatively, one could simply note that the terms with a gluing parameter and no  $\mathcal{T}^{k+1}$  would have to occur in negative index, and this is prohibited by the regularity of  $\mathbf{K}_\alpha$  and  $\mathbf{K}_\beta$ .  $\square$

For  $d \geq 1$ , define  $T^d: CW_0^*(L_{d-1}, L_d) \otimes \dots \otimes CW_0^*(L_0, L_1) \rightarrow CW_1^*(L_0, L_d)$  by

$$(3.28) \quad T^d(\gamma_d, \dots, \gamma_1) = \sum_{\substack{\gamma_0 \in \mathcal{X}_1(L_0, L_1) \\ \deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = -d}} \# \mathcal{T}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0) \cdot \gamma_0.$$

By definition, we may treat  $T$  as a pre-natural transformation from  $\mathcal{F}_\alpha$  to  $\mathcal{F}_\beta$ . The characterization in Lemma 3.31 of the boundary strata of the 1-dimensional components of  $\mathcal{T}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$

implies that

$$\begin{aligned}
& \mathcal{F}_\beta^{d+1}(\gamma_d, \dots, \gamma_1) - \mathcal{F}_\alpha^{d+1}(\gamma_d, \dots, \gamma_1) = \\
& \sum_{m=0}^{d-1} \sum_{i=0}^m T^{m+1} \left( \gamma_d, \dots, \gamma_{i+d-m+1}, \mu_0^{d-m}(\gamma_{i+d-m}, \dots, \gamma_{i+1}), \gamma_i, \dots, \gamma_1 \right) \\
& + \sum_{k=1}^d \sum_{r=1}^k \sum_{\substack{m_i \geq 1 \\ \sum_{i=1}^k m_i = d}} \mu_1^k \left( \mathcal{F}_\beta^{m_k}(\gamma_{\sum_{j=1}^k m_j}, \dots, \gamma_{\sum_{j=1}^{k-1} m_j+1}), \dots, \mathcal{F}_\beta^{m_{r+1}}(\gamma_{\sum_{j=1}^{r+1} m_j}, \dots, \gamma_{\sum_{j=1}^r m_j+1}), \right. \\
& \quad \left. T^{m_r}(\gamma_{\sum_{j=1}^r m_j}, \dots, \gamma_{\sum_{j=1}^{r-1} m_j+1}), \right. \\
& \quad \left. \mathcal{F}_\alpha^{m_{r-1}}(\gamma_{\sum_{j=1}^{r-1} m_j}, \dots, \gamma_{\sum_{j=1}^{r-2} m_j+1}), \dots, \mathcal{F}_\alpha^{m_1}(\gamma_{m_1}, \dots, \gamma_1) \right).
\end{aligned}$$

This collection of equations, as  $d$  ranges over the positive integers, is precisely the condition that  $\mathcal{F}_\beta - \mathcal{F}_\alpha = dT$ , i.e.  $T$  generates an  $A_\infty$ -homotopy between  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$ .

By positivity of intersections, we see as in Lemma 3.5 that  $T^d$  sends  $\mathcal{W}_{\sigma,0}(M, \lambda_M)$  to  $\mathcal{W}_{\sigma,1}(M, \lambda_M)$ , so that  $T$  induces a homotopy between  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  on the partially wrapped Fukaya categories as well.

**3.6. Moving stops.** We are now equipped to prove the following statement.

**Proposition 3.32.** *Suppose  $(M, \lambda_M, \sigma)$  and  $(M', \lambda_{M'}, \sigma')$  are equivalent pumpkin domains. Then  $\mathcal{W}_\sigma(M)$  is quasi-equivalent to  $\mathcal{W}_{\sigma'}(M')$ .*

For this, it is enough to consider the case where  $M = M'$  is a fixed Liouville domain and the equivalence of pumpkin domains arises from an isotopy of stops  $\sigma_t$  with varying fiber. Note that if the fiber were nonvarying, then one could find a path of Liouville isomorphisms  $M \rightarrow M$  which take  $\sigma_0$  to  $\sigma_t$ . For time-dependent fiber, this fails. Since we may decompose the isotopy  $\sigma_t$  as a sequence of small isotopies, Proposition 3.32 follows from the easier

**Lemma 3.33.** *Let  $(M, \lambda_M)$  be a Liouville domain and  $\sigma_t$  a  $[0, 1]$ -parametrized family of pumpkin structures on  $M$ . Suppose every interior Lagrangian in  $(M, \lambda_M, \sigma_0)$  is a Lagrangian in  $(M, \lambda_M, \sigma_t)$  for all  $t$ . Then there is a quasi-equivalence  $\mathcal{W}_{\sigma_0}^{int}(M) \rightarrow \mathcal{W}_{\sigma_1}(M)$ .*

The rest of this section is devoted to the proof of Lemma 3.33. In what follows, we will use  $M_0$  to denote the pumpkin domain  $(M, \lambda_M, \sigma_0)$  and  $M_1$  to denote the pumpkin domain  $(M, \lambda_M, \sigma_1)$ . From the data of the stops, we obtain two presentations of the fully wrapped Fukaya category  $\mathcal{W}(M_0)$  and  $\mathcal{W}(M_1)$ . These categories are quasi-equivalent via any continuation functor, but the quasi-equivalences don't respect the stop filtrations. To obtain a map of partially wrapped Fukaya categories, we will need to be more careful.

The idea is to construct a sequence of continuation functors associated to ever-slower isotopies of the stops for which we will be able to take Gromov limits. Unlike the continuation functors themselves, these Gromov limits will satisfy positivity of intersections with the divisors, which will allow us to define maps of partially wrapped Fukaya categories. To construct our sequence of functors, we will consider Floer data on the multiplihedra compatible with increasingly shifted strip-like ends. To ensure Gromov convergence, we need to carefully choose our isotopies to be compatible as we move along the sequence. For this, we make the following technical definition.

**Definition 3.34.** A **slowing family** consists of the following data.

- For each integer  $n \geq 1$ , a universal choice of strip-like ends  $\epsilon_{\mathcal{S},n}$  for the multiplihedra.
- For each integer  $n \geq 1$  and all  $d \geq 1$ , a diffeomorphism  $\Phi_n^d: \mathcal{S}^{d+1} \rightarrow \mathcal{S}^{d+1}$  and a family of diffeomorphisms  $I_{(\Sigma,w),n}: \Sigma \rightarrow \Phi_n^d(\Sigma)$  parametrized by  $(\Sigma, w) \in \mathcal{S}^{d+1}$ .



- For all  $d \geq 1$  and each  $(\Sigma, w) \in \mathcal{S}^{d+1}$ , a function  $\mathbf{t}_{(\Sigma, w)}: \Sigma \rightarrow [0, 1]$  varying smoothly on  $\mathcal{S}^{d+1}$  with the following properties.

These data are required to satisfy the following conditions.

- (1)  $\Phi_1^d = \text{id}_{\mathcal{S}^{d+1}}$ , and  $\Phi_n^d$  is isotopic to  $\text{id}_{\mathcal{S}^{d+1}}$ .
- (2) For  $(\Sigma, w)$  near the boundary of  $\mathcal{S}^{d+1}$ , let  $(\Sigma_i, w_i)$  be the disks in  $\mathcal{S}^{k_i+1}$  for  $k_i < d$  and  $\Sigma_j$  be the disks in  $\mathcal{R}^{m_j+1}$  for  $m_j \leq d$  from which  $(\Sigma, w)$  is glued. Then  $\Phi_n^d(\Sigma, w)$  is glued from  $\Phi_n^{k_i}(\Sigma_i, w_i)$  and  $\Sigma_j$ .
- (3)  $I_{(\Sigma, w), 1} = \text{id}_\Sigma$ , and  $I_{(\Sigma, w), n}$  is isotopic to  $\text{id}_\Sigma$ . Additionally,  $I_{(\Sigma, w), n}$  sends  $\epsilon_{\mathcal{S}, n}$  to  $\epsilon_{\mathcal{S}, 1}$ .
- (4) For  $(\Sigma, w)$  near the boundary of  $\mathcal{S}^{d+1}$ , let  $(\Sigma_i, w_i)$  be the disks in  $\mathcal{S}^{k_i+1}$  for  $k_i < d$  and  $\Sigma_j$  be the disks in  $\mathcal{R}^{m_j+1}$  for  $m_j \leq d$  from which  $(\Sigma, w)$  is glued. Let  $\Sigma_i^0 \subset \Sigma_i$  be the complement of the strip-like ends. Then under the identifications coming from condition (2), the restrictions to  $\Sigma_i^0$  of  $I_{(\Sigma, w), n}$  and  $I_{(\Sigma_i, w_i), n}$  coincide, and the restriction of  $I_{(\Sigma, w), n}$  to  $\Sigma_j$  is the identity.
- (5) For any fixed  $d$  and  $k$ , the family  $I$  over  $\mathcal{S}^{d+1} \times \mathbb{Z}_{>0}$  is uniformly bounded in  $C^k$ .
- (6) For all  $(\Sigma, w) \in \mathcal{S}^{d+1}$ , there is a decomposition  $\Sigma = U \amalg S$ , where  $U$  is open and  $S$  is biholomorphic to a disjoint union of rectangles  $R_i$ , with the following properties.
  - (a)  $I_{(\Phi_n^d)^{-1}(\Sigma, w), n}$  is holomorphic on  $(I_{(\Phi_n^d)^{-1}(\Sigma, w), n})^{-1}(U)$ .
  - (b)  $R_i$  can be taken to be of the form  $[0, a_i] \times [0, 1]$ , with  $\partial\Sigma \cap S$  mapping to  $[0, a_i] \times \{0, 1\}$ . Moreover,  $(I_{(\Phi_n^d)^{-1}(\Sigma, w), n})^{-1}(R_i)$  is biholomorphic to a rectangle  $[0, b_{i,n}] \times [0, 1]$ , and in these coordinates  $I_{(\Phi_n^d)^{-1}(\Sigma, w), n}(s, t)$  takes the form  $(f_{i,n}(s), t)$ .
  - (c) The above functions  $f_{i,n}$  satisfy  $\frac{\partial f_{i,n}}{\partial s} \leq 1$  everywhere and  $\frac{\partial f_{i,n}}{\partial s} < \frac{1}{n}$  on  $f_{i,n}^{-1}([\frac{a_i}{3}, \frac{2a_i}{3}])$ .
- (7)  $\mathbf{t}_{(\Sigma, w)}$  is 0 on the positive strip-like ends and 1 on the negative strip-like ends.
- (8) For  $(\Sigma, w)$  near the boundary of  $\mathcal{S}^{d+1}$ , let  $(\Sigma_i, w_i)$  be the disks in  $\mathcal{S}^{k_i+1}$  for  $k_i < d$  from which  $(\Sigma, w)$  is glued. Then  $\mathbf{t}_{(\Sigma, w)}$  agrees with the extension by 1 and/or 0 of the functions  $\mathbf{t}_{(\Sigma_i, w_i)}$ .
- (9) In the decomposition of condition (6),  $d\mathbf{t}_{(\Sigma, w)}$  is supported on the union of the subrectangles  $[\frac{a_i}{3}, \frac{2a_i}{3}] \times [0, 1]$ , and on these subrectangles  $\mathbf{t}_{(\Sigma, w)}$  depends only on the first coordinate.

Note that condition (6c) implies that the rectangle lengths  $b_{i,n}$  tend to infinity with  $n$ .

To construct a slowing family, one chooses first  $\epsilon_{\mathcal{S}, 1}$  arbitrarily and second the family of functions  $\mathbf{t}$  such that  $d\mathbf{t}_{(\Sigma, w)}$  is supported near the thin part. Then, one arranges that the embeddings and additional choices of strip-like ends increase the gluing lengths and stretch the support of  $d\mathbf{t}_{(\Sigma, w)}$ .

From the perspective of Floer theory, a slowing family modifies the compatibility conditions between Hamiltonians or almost complex structures and stops. When the stops were fixed, compatibility was roughly the condition that their divisors were almost complex submanifolds. In the presence of a slowing family, we ask that this holds pointwise in the domain. Concretely, let  $\mathbf{t}$  be a family over the multiplihedron of functions  $\mathbf{t}_{(\Sigma, w)}: \Sigma \rightarrow [0, 1]$ . Then a  $\mathbf{t}$ -compatible Hamiltonian on  $(\Sigma, w) \in \mathcal{S}^{d+1}$  is a  $\Sigma$ -parametrized quadratic Hamiltonian  $H$  such that, for all  $z \in \Sigma$  and all  $\sigma \in \sigma_{\mathbf{t}_{(\Sigma, w)}(z)}$ ,  $X_{H(z)}$  is tangent to  $D_\sigma$ . Likewise, an almost complex structure is adapted to  $(\mathbf{t}, H)$  if, for all  $z \in \Sigma$ , it lies in  $\mathcal{J}(M, H(z))$  for the pumpkin structure  $\sigma_{\mathbf{t}_{(\Sigma, w)}(z)}$ .

Let  $\mathbf{K}_0$  and  $\mathbf{K}_1$  be universal and conformally consistent choices of Floer data for the  $A_\infty$  structure for  $M_0$  and  $M_1$ , respectively. Fix a slowing family, and let  $\mathbf{K}^1$  be a universal and conformally consistent choice of Floer data for  $\epsilon_{\mathcal{S}, 1}$  which is regular in the sense of Lemma 3.28. Here, the notion of a universal and conformally consistent choice of Floer data is as for the multiplihedra in Section 3.4, except that the family of Hamiltonians  $H$  is required to be  $\mathbf{t}$ -compatible and the family of almost complex structures  $J$  is required to be adapted to  $(\mathbf{t}, H)$ . Assume moreover that  $H$  depends only on the first coordinate in the rectangles of condition (6).

For  $n \geq 2$ , let  $\mathbf{K}^n$  be a universal and conformally consistent choice of Floer data for  $\epsilon_{\mathcal{S},n}$  which is regular,  $2^{-n}$ -close to  $I_n^* \mathbf{K}^1$  in  $C^n$ , and whose maximum principle kicks in less than 1 later than that of  $\mathbf{K}^1$ . To be more precise, this last condition says that  $R(\mathbf{K}^n) < R(I_n^* \mathbf{K}^1) + 1$ , where  $R$  is the constant of Lemma A.8. This condition is not open, but it is large: it becomes open if we restrict to perturbations of  $H$  supported in  $\{d\beta \leq -c\}$  with  $c > 0$ , and it is nonempty for small  $c$ . Define  $\mathcal{F}_n: \mathcal{W}^{int}(M_0) \rightarrow \mathcal{W}(M_1)$  to be the continuation functor determined by  $\mathbf{K}^n$ .

In addition to functors, we need homotopies of the same kind. For this, we consider the analog of a slowing family for the homotohedron.

**Definition 3.35.** Fix a slowing family. Then an **interpolating family** consists of the following data.

- For each integer  $n \geq 1$ , a universal choice of strip-like ends  $\epsilon_{\mathcal{T},n}$  for the homotohedra.
- For each integer  $n \geq 1$  and all  $d \geq 1$ , a diffeomorphism  $\Psi_n^d: \mathcal{T}^{d+1} \rightarrow \mathcal{T}^{d+1}$  and a family of diffeomorphisms  $I_{(\Sigma,w,q),n}: \Sigma \rightarrow \Psi_n^d(\Sigma)$  parametrized by  $(\Sigma, w, q) \in \mathcal{T}^{d+1}$ .
- For all  $d \geq 1$  and each  $(\Sigma, w, q) \in \mathcal{T}^{d+1}$ , a function  $\mathbf{t}_{(\Sigma,w,q)}: \Sigma \rightarrow [0, 1]$  varying smoothly on  $\mathcal{T}^{d+1}$  with the following properties.

These data are required to satisfy the following conditions.

- (1)  $\epsilon_{\mathcal{T},n}$  interpolates between  $\epsilon_{\mathcal{S},n}$  and  $\epsilon_{\mathcal{S},n+1}$  in the sense of Construction 3.30. That is, it agrees with  $\epsilon_{\mathcal{S},n}$  for small  $q$  and  $\epsilon_{\mathcal{S},n+1}$  for large  $q$ .
- (2)  $\Psi_1^d = \text{id}_{\mathcal{T}^{d+1}}$  and  $\Psi_n^d$  is isotopic to  $\text{id}_{\mathcal{T}^{d+1}}$ . Additionally,  $\Psi_n^d$  fixes the  $q$ -component.
- (3) For  $(\Sigma, w, q)$  near the boundary of  $\mathcal{T}^{d+1}$ , let  $(\Sigma_i, w_i)$  be the disks in  $\mathcal{S}^{k_i+1}$  for  $k_i \leq d$  and  $\Sigma_j$  be the disks in  $\mathcal{R}^{m_j+1}$  for  $m_j \leq d$  and, if applicable,  $(\Sigma', w', q')$  the disk in  $\mathcal{T}^{j+1}$  from which  $(\Sigma, w, q)$  is glued. Then  $\Psi_n^d(\Sigma, w, q)$  is glued from  $\Phi_n^{k_i}(\Sigma_i, w_i)$  for  $(\Sigma_i, w_i)$  of type  $\alpha$ ,  $(\Phi_2^{k_i})^{-1}(\Phi_{n+1}^{k_i}(\Sigma_i, w_i))$  for  $(\Sigma_i, w_i)$  of type  $\beta$ ,  $\Psi_n^j(\Sigma', w', q')$ , and  $\Sigma_j$ . Here, the types  $\alpha$  and  $\beta$  refer to the assignments in (3.24) and (3.25).
- (4)  $I_{(\Sigma,w,q),1} = \text{id}_\Sigma$ , and  $I_{(\Sigma,w,q),n}$  is isotopic to  $\text{id}_\Sigma$ . Additionally,  $I_{(\Sigma,w,q),n}$  sends  $\epsilon_{\mathcal{T},n}$  to  $\epsilon_{\mathcal{T},1}$ .
- (5) For  $(\Sigma, w, q)$  near the boundary of  $\mathcal{T}^{d+1}$ , let  $(\Sigma_i, w_i)$  be the disks in  $\mathcal{S}^{k_i+1}$  for  $k_i \leq d$  and  $\Sigma_j$  be the disks in  $\mathcal{R}^{m_j+1}$  for  $m_j \leq d$  and, if applicable,  $(\Sigma', w', q')$  the disk in  $\mathcal{T}^{j+1}$  from which  $(\Sigma, w, q)$  is glued. Let  $\Sigma_i^0 \subset \Sigma_i$  and  $(\Sigma')^0 \subset \Sigma'$  be the complements of the strip-like ends. Then under the identifications coming from condition (3), the restriction of  $I_{(\Sigma,w,q),n}$  to  $\Sigma_i^0$  coincides with  $I_{(\Sigma_i,w_i),n}$  for  $(\Sigma_i, w_i)$  of type  $\alpha$  and  $(I_{(\Sigma_i,w_i),2})^{-1} \circ I_{(\Sigma_i,w_i),n+1}$  for  $(\Sigma_i, w_i)$  of type  $\beta$ . Similarly, the restrictions to  $(\Sigma')^0$  of  $I_{(\Sigma,w,q),n}$  and  $I_{(\Sigma',w',q'),n}$  coincide. Finally, the restriction of  $I_{(\Sigma,w,q),n}$  to  $\Sigma_j$  is the identity.
- (6) For any fixed  $d$  and  $k$ , the family  $I$  over  $\mathcal{T}^{d+1} \times \mathbb{Z}_{>0}$  is uniformly bounded in  $C^k$ .
- (7) For all  $(\Sigma, w, q) \in \mathcal{T}^{d+1}$ , there is a decomposition  $\Sigma = U \amalg S$ , where  $U$  is open and  $S$  is biholomorphic to a disjoint union of rectangles  $R_i$ , with the following properties.
  - (a)  $I_{(\Psi_n^d)^{-1}(\Sigma,w,q),n}$  is holomorphic on  $(I_{(\Psi_n^d)^{-1}(\Sigma,w,q),n})^{-1}(U)$ .
  - (b)  $R_i$  can be taken to be of the form  $[0, a_i] \times [0, 1]$ , with  $\partial\Sigma \cap S$  mapping to  $[0, a_i] \times \{0, 1\}$ . Moreover,  $(I_{(\Psi_n^d)^{-1}(\Sigma,w,q),n})^{-1}(R_i)$  is biholomorphic to a rectangle  $[0, b_{i,n}] \times [0, 1]$ , and in these coordinates  $I_{(\Psi_n^d)^{-1}(\Sigma,w,q),n}(s, t)$  takes the form  $(f_{i,n}(s), t)$ .
  - (c) The above functions  $f_{i,n}$  satisfy  $\frac{\partial f_{i,n}}{\partial s} \leq 1$  everywhere and  $\frac{\partial f_{i,n}}{\partial s} < \frac{1}{n}$  on  $f_{i,n}^{-1}([\frac{a_i}{3}, \frac{2a_i}{3}])$ .
- (8)  $\mathbf{t}_{(\Sigma,w,q)}$  is 0 on the positive strip-like ends and 1 on the negative strip-like ends.
- (9) For  $(\Sigma, w, q)$  near the boundary of  $\mathcal{T}^{d+1}$ , let  $(\Sigma_i, w_i)$  be the disks in  $\mathcal{S}^{k_i+1}$  for  $k_i \leq d$  and, if applicable,  $(\Sigma', w', q')$  the disk in  $\mathcal{T}^{j+1}$  from which  $(\Sigma, w, q)$  is glued. Then  $\mathbf{t}_{(\Sigma,w,q)}$  agrees with the extension by 1 and/or 0 of the functions  $\mathbf{t}_{(\Sigma_i,w_i)}$  for  $(\Sigma_i, w_i)$  of type  $\alpha$ ,  $(I_{(\Sigma_i,w_i),2})^* \mathbf{t}_{\Phi_2^{k_i}(\Sigma_i,w_i)}$  for  $(\Sigma_i, w_i)$  of type  $\beta$ , and  $\mathbf{t}_{(\Sigma',w',q')}$ .

- (10) In the decomposition of condition (7),  $dt_{(\Sigma, w, q)}$  is supported on the union of the subrectangles  $[\frac{a_i}{3}, \frac{2a_i}{3}] \times [0, 1]$ , and on these subrectangles  $t_{(\Sigma, w, q)}$  depends only on the first coordinate.

Fix an interpolating family. As with functors, choose a universal and conformally consistent family of Floer data  $\mathbf{K}_{\mathcal{T}}^1$  for  $\epsilon_{\mathcal{T}, 1}$  interpolating between  $\mathbf{K}^1$  and  $\mathbf{K}^2$  which is regular in the sense of 3.31 and whose Hamiltonian depends only on the first coordinate in the rectangles of condition (7). For  $n \geq 2$ , let  $\mathbf{K}_{\mathcal{T}}^n$  be a universal and conformally consistent choice of Floer data for  $\epsilon_{\mathcal{T}, n}$  interpolating between  $\mathbf{K}^n$  and  $\mathbf{K}^{n+1}$  which is regular,  $2^{1-n}$ -close to  $I_n^* \mathbf{K}_{\mathcal{T}}^1$  in  $C^n$ , and whose maximum principle kicks in less than 1 later than that of  $\mathbf{K}_{\mathcal{T}}^1$ . Define  $T_n$  to be the pre-natural transformation generating a homotopy from  $\mathcal{F}_n$  to  $\mathcal{F}_{n+1}$  determined by  $\mathbf{K}_{\mathcal{T}}^n$ .

*Proof of Lemma 3.33.* We begin by replacing  $\mathcal{W}(M_1)$  with the limit of the diagram

$$\cdots \xrightarrow{\text{id}} \mathcal{W}(M_1) \xrightarrow{\text{id}} \mathcal{W}(M_1) \xrightarrow{\text{id}} \mathcal{W}(M_1),$$

which can be described in the following way. This is a category  $\mathcal{W}^{lim}(M_1)$  whose objects are the same as those of  $\mathcal{W}(M_1)$  and whose morphism spaces are homotopy limits of the identity chain map. Concretely, this means

$$\text{hom}_{\mathcal{W}^{lim}(M_1)}^k(L_0, L_1) = \prod_{n=1}^{\infty} \text{hom}_{\mathcal{W}(M_1)}^k(L_0, L_1)_n,$$

where

$$\text{hom}_{\mathcal{W}^{lim}(M_1)}^k(L_0, L_1)_n = \text{hom}_{\mathcal{W}(M_1)}^k(L_0, L_1) \oplus \text{hom}_{\mathcal{W}(M_1)}^{k-1}(L_0, L_1)$$

for all  $n$ . Let

$$g = ((\gamma^1, \eta^1), (\gamma^2, \eta^2), \dots) \in \text{hom}_{\mathcal{W}^{lim}(M_1)}(L_0, L_1).$$

The differential  $\mu_{\mathcal{W}^{lim}(M_1)}^1$  is given by the formula

$$\left( \mu_{\mathcal{W}^{lim}(M_1)}^1 g \right)_n = (\partial \gamma^n, \gamma^n + \gamma^{n+1} + \partial \eta^n),$$

where  $\partial = \mu_{\mathcal{W}(M_1)}^1$ . This can be visualized diagrammatically as

$$\begin{array}{ccccc} \cdots & \xrightarrow{\text{id}} & \text{hom}_{\mathcal{W}(M_1)}^*(L_0, L_1) & \xrightarrow{\text{id}} & \text{hom}_{\mathcal{W}(M_1)}^*(L_0, L_1) & \xrightarrow{\text{id}} & \text{hom}_{\mathcal{W}(M_1)}^*(L_0, L_1) \\ & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} \\ & & \text{hom}_{\mathcal{W}(M_1)}^*(L_0, L_1) & \xrightarrow{\text{id}} & \text{hom}_{\mathcal{W}(M_1)}^*(L_0, L_1) & \xrightarrow{\text{id}} & \text{hom}_{\mathcal{W}(M_1)}^*(L_0, L_1) \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \end{array}$$

The higher  $A_{\infty}$  maps are defined via

$$\left( \mu_{\mathcal{W}^{lim}(M_1)}^d(g_d, \dots, d_1) \right)_n = \left( \mu_{\mathcal{W}(M_1)}^d(\gamma_d^n, \dots, \gamma_1^n), \sum_{i=1}^d \mu_{\mathcal{W}(M_1)}^d(\gamma_d^{n+1}, \dots, \gamma_{i+1}^{n+1}, \eta_i^n, \gamma_{i-1}^n, \dots, \gamma_1^n) \right).$$

We will also want the subcategory  $\mathcal{W}_{\sigma_1}^{lim}(M)$  generated by sequences  $g$  of chords which satisfy  $n_{\sigma}(\gamma^i) = n_{\sigma}(\eta^i) = 0$  for all  $\sigma \in \sigma_1$  and all sufficiently large  $i$ . In other words, it consists of those generators whose total intersection number with the stops is finite.

There is a strict  $A_{\infty}$ -functor  $\mathcal{W}(M_1) \rightarrow \mathcal{W}^{lim}(M_1)$  which is the identity on objects and sends  $\gamma \in \text{hom}_{\mathcal{W}(M_1)}(L_0, L_1)$  to  $((\gamma, 0), (\gamma, 0), \dots)$ . This functor is an isomorphism on homology, and so it is a quasi-equivalence. Similarly, the restriction of this functor to  $\mathcal{W}_{\sigma_1}(M)$  is a quasi-equivalence onto  $\mathcal{W}_{\sigma_1}^{lim}(M)$ . Thus, to construct a functor to  $\mathcal{W}_{\sigma_1}(M)$ , it suffices to give a functor to  $\mathcal{W}^{lim}(M_1)$  whose image lies in  $\mathcal{W}_{\sigma_1}^{lim}(M)$ .

Now, the sequences  $\mathcal{F}_n$  and  $T_n$  together define an  $A_\infty$ -functor  $\mathcal{F}: \mathcal{W}^{int}(M_0) \rightarrow \mathcal{W}^{lim}(M_1)$  given by the formula

$$\mathcal{F}^d = ((\mathcal{F}_1^d, T_1^d), (\mathcal{F}_2^d, T_2^d), \dots).$$

We wish to show that  $\mathcal{F}|_{\mathcal{W}_{\sigma_0}^{int}(M)}$  maps into  $\mathcal{W}_{\sigma_1}^{lim}(M)$ . For this, it suffices to consider a fixed stop  $\sigma_t \in \sigma_t$  and fixed inputs and outputs. We want to show that a sequence of holomorphic curves  $u_n \in \mathcal{S}_{\mathbf{K}^n}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$  must eventually have nonnegative topological intersection number with  $D_{\sigma_t}$ , where  $\sigma_t$  is the  $\Sigma$ -parametrized version of  $\sigma_t$ . We also need the same statement with  $\mathcal{T}_{\mathbf{K}_7^n}^{d+1}$  instead of  $\mathcal{S}_{\mathbf{K}^n}^{d+1}$ , but the proof of that is identical.

Assume for a contradiction that there is an increasing sequence  $n_j$  such that  $u_{n_j}$  has strictly negative topological intersection number with  $D_{\sigma_t}$ . Without loss of generality, assume further that all compatible Hamiltonians are split near  $D_{\sigma_t}$ . This is achieved, for example, by the Hamiltonians constructed in the proof of Lemma 2.14. Choose a small  $t$ -dependent tubular neighborhood of  $D_{\sigma_t}$ , and let  $S$  be the boundary of that neighborhood. For a generic choice of  $S$ ,  $u_{n_j}^{-1}(S)$  is a disjoint union of embedded circles for all  $j$ . Let  $\Sigma_{n_j}$  be the domain of  $u_{n_j}$ , and let  $z_{n_j} \in \Sigma_{n_j}$  be such that  $u_{n_j}(z_{n_j}) \in D_{\sigma_t(z_j)}$  and  $u_{n_j}$  has negative degree on the smallest circle in  $u_{n_j}^{-1}(S)$  surrounding  $z_{n_j}$ .

Since all Floer data are approximately pulled back from fixed ones on a compact domain, Arzelà-Ascoli applies to give a subsequence  $n_{j'}$  for which  $\mathbf{K}^{n_{j'}}$  and the complex structures on the domain converge in  $C^\infty$  on an increasing sequence of neighborhoods of  $z_{n_{j'}}$ . The limit of this data is a boundary-punctured Riemann surface  $\Sigma$  together with a Hamiltonian  $H$ , a sub-closed 1-form  $\beta$ , and an almost-complex structure  $J \in \mathcal{J}(M, H)$  which are compatible with the stop  $\sigma_t$ , where  $t$  is now constant. The rectangular coordinates from the definition of a slowing family give rise to strip-like ends on  $\Sigma$ , and in these coordinates  $\beta$  is asymptotic to a 1-form of the form  $f(t)dt$ .

Now the Floer data were chosen to satisfy a uniform maximum principle, and hence we may apply Gromov compactness at  $z_{n_{j'}}$  to get a further subsequence  $u_{n_{j''}}$  which converges in  $C_{loc}^\infty$  to some nonconstant curve  $u$  with domain  $\Sigma$ . There is no issue of bubbling at  $z = \lim z_{n_{j''}}$  because the symplectic form is exact. Now  $t$  is constant on  $\Sigma$ , so positivity of intersections applies. In particular,  $u$  has only positive intersections with  $D_{\sigma_t}$ , and  $u(z)$  is such an intersection. Since  $u$  has finite energy, it is asymptotic at the punctures to  $X_H$ -chords, and because  $H$  is split such chords cannot intersect  $S$ . Thus,  $u^{-1}(S)$  separates  $z$  from  $\partial\Sigma$ .

Identifying  $z_{n_{j''}}$  and its neighborhoods with  $z$  and subsets of  $\Sigma$ , we see now that the smallest circle of  $u_{n_{j''}}^{-1}(S)$  surrounding  $z$  is contained in  $\Sigma$  for large  $j''$ . Since  $u$  is  $C^0$ -close to  $u_{n_{j''}}$ , it has negative winding number about  $D_{\sigma_t}$  on this circle. This contradicts positivity of intersections for  $u$  and completes the proof.  $\square$

#### 4. NONDEGENERATE STOPS

**4.1. Action filtrations for Hochschild homology.** To define a strongly nondegenerate stop, we will need to transfer the action filtration from wrapped Floer homology to Hochschild homology. Unfortunately, the energy leaks of Lemma A.2 prevent the action from extending directly to a filtration on the Fukaya category, so we will need to find a way to package those leaks.

To begin, recall that for any  $A_\infty$ -category  $\mathcal{A}$ , the Hochschild homology of  $\mathcal{A}$  can be given as the homology of a chain complex

$$(4.1) \quad CC_*(\mathcal{A}) = \bigoplus_{d=1}^{\infty} \bigoplus_{\text{words}} \mathbb{K} \gamma_d \otimes \dots \otimes \gamma_1$$

where  $\gamma_i \in \text{hom}(L_i, L_{i+1})$  is a cyclically composable sequence of morphisms in  $\mathcal{A}$ . The grading is cohomological and is given by

$$\deg(\gamma_d \otimes \cdots \otimes \gamma_1) = \sum_{i=1}^d \deg(\gamma_i) + 1 - d.$$

The differential  $\delta: CC_*(\mathcal{A}) \rightarrow CC_{*+1}(\mathcal{A})$  comes from the  $A_\infty$  structure on  $\mathcal{A}$ , namely

$$\begin{aligned} \delta(\gamma_d \otimes \cdots \otimes \gamma_1) &= \sum_{\substack{i,j \geq 0 \\ i+j < d}} \mu^{i+j+1}(\gamma_i, \dots, \gamma_{d-j}) \otimes \gamma_{d-j-1} \otimes \cdots \otimes \gamma_{i+1} \\ &+ \sum_{\substack{i,j \geq 1 \\ i+j \leq d}} \gamma_d \otimes \cdots \otimes \gamma_{i+j} \otimes \mu^j(\gamma_{i+j-1}, \dots, \gamma_i) \otimes \gamma_{i-1} \otimes \cdots \otimes \gamma_1 \end{aligned}$$

where  $\gamma_0 := \gamma_d$ .

Now let  $(F, \lambda_F)$  be a Liouville domain, which we think of as the fiber of a stop. Choose an auxiliary metric on  $\hat{F}$  which  $\hat{Z}_F$ -invariant outside of a compact set. A **normalizing Hamiltonian** is a function  $H^b: \hat{F} \rightarrow \mathbb{R}_{\geq 0}$  which is smooth and satisfies

- (1)  $H^b = Q\kappa(Q)$ , where
  - (a)  $Q: \hat{F} \rightarrow \mathbb{R}_{\geq 0}$  is a proper, strictly quadratic continuous function.
  - (b)  $\kappa: \mathbb{R}_{\geq 0} \rightarrow [0, 1]$  is a nondecreasing cutoff function which is 0 in a neighborhood of 0 and is 1 outside a compact set.
- (2)  $\|H^b\|_{C^2} \ll 1$  whenever  $dH^b(Z) \neq 2H^b$ . Note that this can be achieved by multiplying any function of the above form by a small positive constant.

*Remark 4.1.* Condition (1) implies that  $dH^b(\hat{Z}) = 2H^b$  outside a compact set, and that  $H_\tau^q \geq \tau H^q$  globally for any  $\tau \geq 1$ . Moreover, given any other  $H^{b'}$  of this form, there is some  $\tau$  such that  $(H^{b'})_\tau \geq H^b$  globally.

Let  $H^b$  be a normalizing Hamiltonian. As the name suggests, we will use it as a normalization for our action filtration. For  $\varepsilon > 0$ , consider the class of functions  $\mathcal{H}_\varepsilon(F)$  consisting of those compatible Hamiltonians  $H$  on  $\hat{F}$  which satisfy

$$(4.2a) \quad \left\| \log \frac{H}{H^b} \right\|_{C^1} < \frac{\varepsilon}{2} \quad \text{whenever} \quad H^b > 1$$

$$(4.2b) \quad \|H - H^b\|_{C^2} < \frac{\varepsilon}{2} \quad \text{whenever} \quad H^b \leq 1$$

These conditions imply that there is a constant  $A_{\min} > 0$  such that for sufficiently small  $\varepsilon$ , any  $H \in \mathcal{H}_\varepsilon(F)$  satisfies

$$(4.3) \quad \text{Every nonconstant time 1 orbit of } X_H \text{ has action greater than } A_{\min}.$$

On the other hand, it is always the case that a constant orbit has negative action.

Consider now a presentation  $\mathcal{W}^\varepsilon(F)$  of the wrapped Fukaya category  $\mathcal{W}(F)$  which satisfies the following conditions.

- (1) The objects consist of all Lagrangians in the sense of Definition 3.1 on which a primitive of  $\hat{\lambda}_F$  is globally constant outside a compact set has total variation bounded by  $\varepsilon$ . Note that every Lagrangian is isotopic to such a Lagrangian via a combination of Moser flow to fix the ends and inverse Liouville flow to reduce the variation.
- (2) For each pair  $(L_0, L_1)$  of Lagrangians, the Hamiltonian  $H^{0,1}$  belongs to  $\mathcal{H}_\varepsilon(F)$ .

- (3) All Floer data for  $\Sigma^{d+1}$  (cf. Definition 3.8) have the following three properties. First,  $e^{1-d}\tau_0 < \tau_E < \tau_0$ . Second, each boundary component  $E$  of  $\Sigma$  contains disjoint open intervals  $E^+$  and  $E^-$  such that  $\tau_E$  is nonincreasing outside  $E^+$  and nondecreasing outside  $E^-$ . Third

$$\int_{\Sigma^{d+1}} \max_{\hat{F}}(d^\Sigma H \wedge \beta) < \frac{2de^{d-1}\varepsilon}{\tau_0}.$$

In the last item, the first two conditions say that the Floer data doesn't rescale excessively, while the last says that it doesn't wobble too much on the compact part of  $\hat{F}$ . To see that this can be achieved, note first that the thick-thin decomposition of  $\Sigma^{d+1}$  has at most  $2d$  thin pieces aside from the output strip-like end. We can choose the Floer data so that  $d^\Sigma H$  is supported near the thin part, where near each piece, as one moves towards the negative end,  $H$  changes within a rescaled  $\mathcal{H}_\varepsilon(F)$  while experiencing a further small positive rescaling. The extent to which it can decrease on the interior is bounded by  $a\varepsilon$ , for some  $a$  slightly greater than 1, times a rescaling factor which depends on the location in  $\Sigma^{d+1}$ . By the first inequality, this local factor is bounded by  $e^{d-1}\tau_0$ .

One could reduce the exponential part of the coefficient from  $e^{d-1}$  to  $(2 + \varepsilon')^{d-1}$  for some small  $\varepsilon'$  depending on  $\varepsilon$ , but conformal consistency prevents us from doing any better than that. This is because the bijection from  $X_H$ -chords to  $X_{H_\tau}$ -chords multiplies action by  $\frac{1}{\tau}$ , and  $\mathcal{R}^{d+1}$  has corners which come from sequentially gluing  $d - 1$  triangles. Each of these triangles rescales the Floer data by a factor of at least 2.

The above conditions allow us to write down a **shifted action**

$$A^\varepsilon : CC_*(\mathcal{W}^\varepsilon(F)) \rightarrow \mathbb{R}_+$$

given by

$$(4.4) \quad A^\varepsilon(\gamma_d \otimes \cdots \otimes \gamma_1) := e^{d-1} \sum_{i=1}^d (A(\gamma_i) + 8\varepsilon)$$

and

$$A^\varepsilon(\vec{\gamma} + \vec{\gamma}') = \max\{A^\varepsilon(\vec{\gamma}), A^\varepsilon(\vec{\gamma}')\}.$$

Here,  $A(\gamma)$  is taken with respect to the Floer data defining the Floer cochain complex (i.e.  $\tau = 1$ ). Even though the action of a chord as defined in Appendix A.1 depends on the choice of a primitive of  $\hat{\lambda}_M$  for each Lagrangian, the shifted action doesn't because it cancels on the cyclic chain.

**Lemma 4.2.**  *$A^\varepsilon$  strictly filters the Hochschild chain complex.*

*Proof.* The condition on the total variation of primitives of Lagrangians, ensures that the primitive contributes no less than  $-\varepsilon$  to the action of a chord, while the definition of  $\mathcal{H}_\varepsilon(F)$  ensures that

$$A_0(\gamma) > -\frac{\varepsilon}{2}$$

for any Floer generator  $\gamma$ . This implies that each term of the sum in (4.4) is greater than  $\frac{13}{2}\varepsilon$ , and in particular it's positive.

Next, the discussion in Appendix A.1 shows that the unshifted action

$$A(\gamma_d \otimes \cdots \otimes \gamma_1) := e^{d-1} \sum_{i=1}^d A(\gamma_i)$$

filters the Hochschild complex up to a leak bounded by

$$\int_{\Sigma} \max_{\hat{F}} d^\Sigma H \wedge \beta + \|f_L\|_{C^0} \int_{\partial\Sigma} \frac{|d\tau_E|}{\tau_E^2}.$$

The conditions on the Floer data defining  $\mathcal{W}^\varepsilon(F)$  ensure that, after normalizing  $\tau_0 = 1$ , this is bounded by  $4de^{d-1}\varepsilon$ .  $\square$

For a homology class  $c \in HH_*(\mathcal{W}^\varepsilon(F))$ , we define its **length- $k$  shifted action** to be

$$A^{\varepsilon,k}(c) = \inf \left\{ A^\varepsilon(\vec{\gamma}) \mid [\vec{\gamma}] = c \text{ and } \vec{\gamma} \in \bigoplus_{i=1}^k \bigoplus_{\text{words}} \mathbb{K} \gamma_i \otimes \cdots \otimes \gamma_1 \right\}.$$

Now any functor between  $A_\infty$ -categories induces a map on Hochschild homology, and hence we get for any continuation functor  $\mathcal{F}: \mathcal{W}^\varepsilon(F) \rightarrow \mathcal{W}(F)$  a homomorphism of graded vector spaces

$$HH(\mathcal{F}): HH_*(\mathcal{W}^\varepsilon(F)) \rightarrow HH_*(\mathcal{W}(F)).$$

From the existence of homotopies between continuation functors, it follows that there is a *canonical* isomorphism  $HH_*(\mathcal{W}^\varepsilon(F)) \cong HH_*(\mathcal{W}(F))$ . Identifying Hochschild homology for the various presentations, we define

**Definition 4.3.** Let  $c \in HH_*(\mathcal{W}(F))$ . Its **length- $k$   $H^b$ -normalized action** is defined to be

$$(4.5) \quad A^{b,k}(c) = \liminf_{\varepsilon \rightarrow 0} \{ A^{\varepsilon,k}(c) \text{ for some presentation of } \mathcal{W}^\varepsilon(F) \}.$$

With the normalized action in hand, we can state the nondegeneracy criterion.

**Definition 4.4.** A Liouville domain  $F$  is called **strongly nondegenerate** if following conditions hold.

- (1) For every connected component  $F_i$  of  $F$ ,  $SH^*(F_i) \neq 0$ . Here,  $SH^*(F)$  is symplectic cohomology, first defined in [16, 10]. See Section 4.3 for a definition.
- (2)  $F$  is nondegenerate, i.e. the unit  $\mathbf{1} \in SH^*(F)$  is in the image of the standard homological open-closed map

$$\mathcal{OC}: HH_*(\mathcal{W}(F)) \rightarrow SH^*(F).$$

Ganatra [19] showed that this implies that  $\mathcal{OC}$  is an isomorphism.

- (3) There is some  $k \in \mathbb{N}$  and some normalizing Hamiltonian  $H^b$  for which  $A^{b,k}(e) = 0$ , where  $e = \mathcal{OC}^{-1}(\mathbf{1})$  is the **Hochschild fundamental class**.

A stop is called **strongly nondegenerate** if its fiber is strongly nondegenerate.

*Example 4.5.* Abouzaid's description of the wrapped Fukaya category of a cotangent bundle [5, 1] shows that any cotangent bundle  $F = T^*X$  is strongly nondegenerate. In this case  $k = \dim(X) + 1$ , and a low-action, length- $k$  Hochschild fundamental cycle (i.e. a representative of  $e$ ) is obtained by picking a fine triangulation of  $X$ . Likewise, for  $F$  a punctured Riemann surface, a low-action, length-3 fundamental cycle can be obtained from a fine lamination of  $F$ . These examples are combined and generalized in the upcoming paper [20], where Ganatra–Pardon–Shende show that any Weinstein domain admitting a skeleton with arboreal singularities [30] is strongly nondegenerate.

As with the  $A_\infty$  operations, the energy leakage of the  $d$ 'th term of a continuation functor between presentations of  $\mathcal{W}^\varepsilon(F)$  can be arranged to be bounded by a constant times  $de^{d-1}\varepsilon$ . For a low-action representative of  $e \in HH_*(\mathcal{W}^\varepsilon(F))$ , this quantity is small because  $d$  is a priori bounded, and thus continuation functors map low-action fundamental cycles to low-action fundamental cycles. Hence, we see that the condition  $A^{b,k}(e) = 0$  can be detected by *any* sequence of presentations of  $\mathcal{W}^\varepsilon(F)$  with  $\varepsilon$  tending to 0.

In fact, in view of Remark 4.1, the above argument shows

**Lemma 4.6.** *If a Liouville domain is strongly nondegenerate with  $A^{b,k}(e) = 0$  for a single normalizing Hamiltonian, then  $A^{b,k}(e) = 0$  for every normalizing Hamiltonian.*  $\square$

**Corollary 4.7.** *In the above situation, given any normalizing Hamiltonian and any sequence of presentations  $\mathcal{W}^{\varepsilon_i}(F)$  for that Hamiltonian with  $\varepsilon_i \rightarrow 0$ , one has  $\lim_{i \rightarrow \infty} A^{\varepsilon_i,k}(e) = 0$ .*  $\square$

**Definition 4.8.** We will refer to the smallest  $k$  such that  $A^{b,k}(e) = 0$  as the **Hochschild length** of  $F$ .

**4.2. Statement of the theorem.** To make Theorem 1.1 precise, we need one more notion:

**Definition 4.9.** Let  $(M, \lambda_M, \sigma)$  be a pumpkin domain, and let  $\sigma \in \sigma$  be a stop with fiber  $F$ . Note that interior Lagrangians of  $M$  and  $\Sigma F$  give rise to interior Lagrangians of  $M[\sigma]$  via inclusion. An interior Lagrangian  $L$  in  $M[\sigma]$  is said to be **supported in  $\sigma$**  if it is isomorphic in  $\mathcal{W}_\sigma(M[\sigma])$  to an interior Lagrangian of  $\Sigma F$ . Let  $\mathcal{B}(\sigma) \subset \mathcal{W}(M[\sigma])$  and  $\mathcal{B}_\sigma(\sigma) \subset \mathcal{W}_\sigma(M[\sigma])$  denote the full subcategories of objects supported in  $\sigma$ . More generally, for a subset  $\sigma' \subset \sigma$ , let  $\mathcal{B}_{\sigma'}(\sigma)$  denote the full subcategory of  $\mathcal{W}_{\sigma'}(M[\sigma])$  composed of objects supported in  $\sigma$ .

Although we will generally work in  $M[\sigma]$  when dealing with a given stop  $\sigma$ , we will want to state results in  $M$  alone. In this case, we abuse notation and denote again by  $\mathcal{B}_\sigma(\sigma) \subset \mathcal{W}_\sigma(M)$  the full subcategory of objects whose image under the quasi-equivalence  $\mathcal{W}_\sigma(M) \rightarrow \mathcal{W}_\sigma(M[\sigma])$  lie in  $\mathcal{B}_\sigma(\sigma)$ .

Note that  $\mathcal{B}(\sigma)$  is a contractible subcategory, in the sense that for all  $L \in \mathcal{B}(\sigma)$ , the unit  $e_L \in \text{hom}_{\mathcal{W}(M)}^0(L, L)$  is exact. To see this, assume  $L \subset M[\sigma]$  is an interior Lagrangian of  $\Sigma F$ . Then that  $L$  can be isotoped clockwise through  $D_\sigma$  to a Lagrangian  $L'$  such that  $L$  has no chords to  $L'$  of small action. This means, by an energy argument, that the isomorphism from  $L$  to  $L'$  given by Lemma 3.23 is the zero morphism, which implies that  $e_L$  is exact. Hence, the same holds for any object isomorphic to  $L$ . More generally, for  $\sigma' \subset \sigma$  not containing  $\sigma$ ,  $\mathcal{B}_{\sigma'}(\sigma)$  is contractible.

Let  $\sigma' = \sigma \setminus \{\sigma\}$ . By the universal property of a quotient category [12, 27], since the image of  $\mathcal{B}_\sigma(\sigma)$  is contractible, the inclusion  $\mathcal{W}_\sigma(M) \rightarrow \mathcal{W}_{\sigma'}(M)$  factors up to homotopy through the quotient  $\mathcal{W}_\sigma(M)/\mathcal{B}_\sigma(\sigma)$ :

$$(4.6) \quad \begin{array}{ccc} & \mathcal{W}_\sigma(M)/\mathcal{B}_\sigma(\sigma) & \\ \nearrow & & \searrow \text{SR} \\ \mathcal{W}_\sigma(M) & \xrightarrow{\quad} & \mathcal{W}_{\sigma'}(M) \end{array}$$

With this, the precise statement of Theorem 1.1 is

**Theorem 4.10.** *Let  $(M, \lambda_M, \sigma)$  be a pumpkin domain, and let  $\sigma \in \sigma$  be a strongly nondegenerate stop. Set  $\sigma' = \sigma \setminus \{\sigma\}$ . Then the map  $\text{SR}: \mathcal{W}_\sigma(M)/\mathcal{B}_\sigma(\sigma) \rightarrow \mathcal{W}_{\sigma'}(M)$  from (4.6) is fully faithful.*

**4.3. Closed strings.** The proof of Theorem 4.10 relies on the existence of a closed string version of partially wrapped Floer homology. This will take the form of a filtration on the symplectic homology chain complex, which we now explain.

Let  $(M, \lambda_M, \sigma)$  be a pumpkin domain. Pick a compatible Hamiltonian  $\tilde{H}$  and an  $S^1$ -family of perturbing Hamiltonians  $P_t: \hat{M} \rightarrow \mathbb{R}_{\geq 0}$ , where  $t \in \mathbb{R}/\mathbb{Z} \cong S^1$ , which satisfy the following conditions.

$$(4.7) \quad \begin{array}{l} P_t \text{ is bounded, and } \|X_{P_t}\| \text{ decays exponentially in the symplectization coordinate } \\ \sqrt{\tilde{H}} \text{ for any metric of the form } \hat{\omega}_M(\cdot, J \cdot) \text{ with } J \in \mathcal{J}(M, H). \end{array}$$

$$(4.8) \quad \begin{array}{l} H_t := \tilde{H} + P_t \text{ is **nondegenerate** in the sense that for any 1-periodic orbit } x \text{ of the} \\ \text{time-dependent vector field } X_{H_t}, \text{ the Poincaré return map of } x \text{ does not have 1 as} \\ \text{an eigenvalue.} \end{array}$$

$$(4.9) \quad \begin{array}{l} \text{For each } \sigma \in \sigma, \tilde{H} \text{ is of the form (2.6) near } D_\sigma, \text{ with } f(z) = c|z|^2 \text{ and } c > 0. \text{ Sim-} \\ \text{ilarly, } P_t \text{ is independent of the } \mathbb{H}_\rho\text{-coordinate near } D_\sigma, \text{ and } X_{H_t} \text{ satisfies condition} \\ \text{(4) of Definition 2.13.} \end{array}$$

Note that perturbing Hamiltonians  $P$  can be constructed by taking sums of functions of the form  $\kappa \circ H$ , where  $H$  is a compatible Hamiltonian and  $\kappa: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a positive nondecreasing function which is eventually constant.



Let  $\mathcal{X}(H_t)$  be the space of 1-periodic orbits of  $H_t$ . Because  $H_t$  is nondegenerate, this is a discrete space. The symplectic cochain complex  $SC^*(M)$  is the graded  $\mathbb{K}$ -vector space generated by  $\mathcal{X}(H_t)$  with grading given by the cohomological Conley-Zehnder index, see [4]. For this situation, we switch to a different convention for almost complex structures. Namely, we follow Ganatra [19] and define the space of almost complex structures of **rescaled contact type**  $\mathcal{J}_{resc}^{S^1}(M, H_t)$ . This consists of  $S^1$ -families of almost complex structures  $J_t$  which are  $\hat{\omega}_M$ -compatible and satisfy the following three conditions. First, there is some  $t$ -independent constant  $c_{resc} > 0$  such that, for all  $t$ ,

$$(4.10) \quad d\tilde{H} \circ J_t = -c_{resc} \hat{\lambda}_M$$

outside of a  $t$ -independent compact set. Second, the restriction  $J_t|_{\ker dH \cap \ker \hat{\lambda}_M}$  is asymptotically  $\hat{Z}_M$ -invariant. Third, for each stop  $\sigma \in \sigma$ , the projection to  $\mathbb{H}_\rho$  is holomorphic along  $D_\sigma$ . Given  $J_t \in \mathcal{J}_{resc}^{S^1}(M, H_t)$ , we can consider maps  $u: \mathbb{R} \times S^1 \rightarrow \hat{M}$  satisfying Floer's equation

$$\partial_s u + J_t(\partial_t u - X_{H_t}) = 0$$

and asymptotic as  $s \rightarrow \pm\infty$  to orbits  $x_\pm \in \mathcal{X}(H_t)$ . The moduli space of such maps, denoted  $\tilde{\mathcal{Q}}(x_+, x_-)$ , satisfies the obvious analog of Lemmas 3.3 and 3.4, i.e.

**Lemma 4.11.** *For generic choices of  $P_t$ , there is a comeager subset*

$$\mathcal{J}_{reg}^{S^1}(M, H_t) \subset \mathcal{J}_{resc}^{S^1}(M, H_t)$$

*such that, for any  $J_t \in \mathcal{J}_{reg}^{S^1}(M, H_t)$  and  $x_\pm \in \mathcal{X}(H_t)$ , the following hold.*

- (1)  $\tilde{\mathcal{Q}}(x_+, x_-)$  is a smooth manifold of dimension  $\deg(x_-) - \deg(x_+)$ , and the translation  $\mathbb{R}$ -action on  $\tilde{\mathcal{Q}}(x_+, x_-)$  is free if and only if  $x_+ \neq x_-$ . In this case, write  $\mathcal{Q}(x_+, x_-)$  for the quotient  $\tilde{\mathcal{Q}}(x_+, x_-)/\mathbb{R}$ .
- (2) If  $\deg(x_-) - \deg(x_+) = 1$ , then  $\mathcal{Q}(x_+, x_-)$  is compact.
- (3) If  $\deg(x_-) - \deg(x_+) = 2$ , then  $\mathcal{Q}(x_+, x_-)$  admits a Gromov compactification as a topological 1-manifold with boundary, and its boundary is in natural bijection with the once-broken configurations  $\coprod_{y \in \mathcal{X}(H_t)} \mathcal{Q}(y, x_-) \times \mathcal{Q}(x_+, y)$ .

*Remark on proof.* Because (4.10) is so stringent, we allow small perturbations of  $P_t$ . These can be made without changing the set  $\mathcal{X}(H_t)$ , and in concert with the freedom to perturb  $c_{resc}$  they allow us to achieve transversality even when  $\dim M = 2$ . This was not an issue for chords because, when  $\dim M = 2$ , all chords outside of a compact set in a given end live in different relative homotopy classes.

With regards to compactness, our maximum principle does not apply in the presence of a time-dependent perturbing Hamiltonian, but by choosing a symplectization coordinate  $r = \sqrt{\tilde{H}}$  we find ourselves in Ganatra's setup and can apply Theorem A.1 of [19].  $\square$

Fix  $J_t \in \mathcal{J}_{reg}^{S^1}(M, H_t)$ . The differential  $\partial$  on  $SC^*(M)$  is given by

$$\partial x_+ = \sum_{\deg(x_-) - \deg(x_+) = 1} \# \mathcal{Q}(x_+, x_-) \cdot x_-$$

and satisfies  $\partial^2 = 0$  by the usual argument which looks at ends of 1-dimensional moduli spaces. The cohomology  $SH^*(M) := H^*(SC^*(M), \partial)$  is known as symplectic cohomology.

The pumpkin structure  $\sigma$  endows  $SC^*(M)$  with a filtration similar to that for open strings but slightly more subtle due to the fact that orbits can live on the divisor of a stop. We describe a part of it, which will suffice for our purposes. Let  $\mathcal{X}_\sigma(H_t) \subset \mathcal{X}(H_t)$  be the set of orbits which do not intersect  $\sigma(\hat{F} \times \mathbb{R}_+)$  and do not live in  $D_\sigma$  for any  $\sigma \in \sigma$ . Let  $SC_\sigma^*(M) \subset SC^*(M)$  be the graded linear subspace generated by  $\mathcal{X}_\sigma(H_t)$ .

**Lemma 4.12.**  $SC_\sigma^*(M)$  is a subcomplex of  $SC^*(M)$ .

*Proof.* We need to show that if  $x_-$  intersects  $\sigma(\hat{F} \times \mathbb{R}_+)$  or lives in  $D_\sigma$  for some  $\sigma \in \sigma$ , then  $\tilde{Q}(x_+; x_-)$  is empty for any  $x_+ \in \mathcal{X}(H_t)$ . In the first case, the conclusion follows from positivity of intersections as in Lemma 3.5. In the second, the asymptotics in [34], combined with assumption (4.9), ensure that  $x_-$  behaves as if it had strictly positive winding number around  $D_\sigma$ . This puts us back in the regime where we can use positivity of intersections.  $\square$

**Definition 4.13.**  $SC_\sigma^*(M)$  is called the **partially wrapped symplectic cochain complex**.

We will be interested in holomorphic curves which interpolate between the open and the closed string worlds. For this, we make the following definitions

**Definition 4.14.** A **punctured Riemann surface with boundary, ends, and cylinders** is a Riemann surface

$$\Sigma = \bar{\Sigma} \setminus (Z_{\partial\Sigma} \cup Z_\Sigma),$$

where  $\bar{\Sigma}$  is a compact Riemann surface with boundary,  $Z_{\partial\Sigma}$  is a finite subset of the boundary of  $\bar{\Sigma}$ , and  $Z_\Sigma$  is a finite subset of the interior of  $\bar{\Sigma}$ , together with the following additional data.

- (1) For each  $\zeta \in Z_{\partial\Sigma}$ , a positive or negative strip-like end at  $\zeta$ .
- (2) For each  $\zeta \in Z_\Sigma$ , a positive or negative cylindrical end at  $\zeta$ . These are holomorphic embeddings

$$(4.11) \quad \epsilon_+ : \mathbb{R}_{\geq 0} \times S^1 \rightarrow \Sigma \quad \text{or} \quad \epsilon_- : \mathbb{R}_{\leq 0} \times S^1 \rightarrow \Sigma,$$

respectively, such that  $\lim_{s \rightarrow \pm\infty} \epsilon_\pm(s, t) = \zeta$ .

- (3) A finite number of **finite cylinders**  $\delta_i$ . These are holomorphic embeddings

$$\delta_i : [a_i, b_i] \times S^1 \rightarrow \text{int}(\Sigma).$$

Additionally, we require that all ends and finite cylinders have disjoint images. For cylindrical ends and finite cylinders, we define their **m-shifts** as with strips and define the **thin part** of  $\Sigma$  to be the union of the 3-shifts of all ends, finite cylinders, and, if  $\Sigma$  comes with an implicit gluing decomposition, finite strip-like gluing regions.

A **punctured Riemann surface with labeled boundary, ends, and cylinders** is a punctured Riemann surface  $\Sigma$  with boundary, ends, and cylinders, along with an assignment of a Lagrangian  $L_i \subset \hat{M}$  to each boundary component  $\partial_i \Sigma$  of  $\Sigma$ .

**Definition 4.15.** Let  $\Sigma$  be a punctured Riemann surface with labeled boundary, ends, and cylinders. A **Floor datum** on  $\Sigma$  is a 5-tuple  $(\beta, H^{strict}, P, J, \tau_E)$ , where

- $\beta$  is a 1-form on  $\Sigma$
- $H^{strict}$  is a  $\Sigma$ -parametrized compatible Hamiltonian
- $P$  is a function  $P : \Sigma \times \hat{M} \rightarrow \mathbb{R}_+$
- $J$  is a  $\Sigma$ -parametrized  $\hat{\omega}_M$ -compatible almost complex structure
- $\tau_E$  is a function  $\tau_E : \partial\Sigma \rightarrow \mathbb{R}_+$

with the following properties.

- (1) Outside the images of the cylindrical ends and finite cylinders,  $(\beta, H^{strict}, J, \tau_E)$  satisfy the conditions of Definition 3.8.
- (2)  $d\beta$ ,  $d^\Sigma H^{strict} \wedge \beta$ , and  $d^\Sigma P \wedge \beta$  are nonpositive everywhere.
- (3) For each cylindrical end  $\epsilon_i$ ,  $(\epsilon_i^1)^* \beta = w_i dt$  for some positive real number  $w_i$ . Similarly, for each finite cylinder  $\delta_i$ ,  $(\delta_i^1)^* \beta = w_i dt$  for some positive real number  $w_i$ .
- (4) For each cylindrical end or finite cylinder, there is a scaling constant  $\tau_i > 0$  such that

$$w_i H^{strict} = \tilde{H}_{\tau_i}$$

on the image of that cylindrical end or finite cylinder.

- (5) There is some strictly positive function  $g: \Sigma \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$dH^{strict} \circ J = -g(H^{strict})\hat{\lambda}_M$$

outside a  $\Sigma$ -independent compact set.

- (6) The restriction  $J|_{\ker dH^{strict} \cap \ker \hat{\lambda}_M}$  is asymptotically  $\hat{Z}_M$ -invariant.  
 (7) For each stop  $\sigma \in \boldsymbol{\sigma}$ , the projection to  $\mathbb{H}_\rho$  is holomorphic along  $D_\sigma$ .  
 (8) For each cylindrical end or finite cylinder,

$$J(s, t) = (J_t)_{\tau_i} := (\phi^{\tau_i})^* J_t$$

in the 2-shift of that cylindrical end or finite cylinder.

- (9)  $P$  is globally bounded, and  $\|X_P\|$  decays exponentially in the symplectization coordinate. Moreover,  $P$  is locally constant outside the 2-shifts of the cylindrical ends and finite cylinders, and in the 3-shifts of the cylindrical ends and finite cylinders it satisfies

$$w_i P(s, t) = (P_t)_{\tau_i} + A_i := \frac{1}{\tau_i} (\phi^{\tau_i})^* P_t + A_i$$

for some constant  $A_i$  depending on the cylindrical region.

For simplicity, define  $H := H^{strict} + P$ .

A Floer datum for a punctured Riemann surface  $\Sigma$  with boundary, ends, cylinders, but no Lagrangian labels, consists of a Floer datum for each Lagrangian labeling of  $\Sigma$ .

For such Floer data, there is again a notion of conformal equivalence. Namely, two Floer data  $(\beta, H^{strict}, P, J, \tau_E)$  and  $(\beta', (H^{strict})', P', J', \tau'_E)$  are **conformally equivalent** if there are constants  $A, C, W$  with  $C, W > 0$  such that

$$\beta = W\beta', \quad H^{strict} = \frac{1}{W}((H^{strict})')_C, \quad P = \frac{1}{W}(P')_C + A, \quad J = (J')_C, \quad \tau_E = C\tau'_E.$$

Compared to the situation without cylinders, we now have the additional freedom to shift  $H$  by a constant.

As before, solutions  $u: \Sigma \rightarrow \hat{M}$  to

$$(4.12) \quad J \circ (du - X_H \otimes \beta) = (du - X_H \otimes \beta) \circ j$$

with boundary conditions  $u(\partial_i E) \subset (\phi^{\tau_E})^* L_i$  are related to solutions  $u': \Sigma \rightarrow \hat{M}$  to

$$(4.13) \quad J' \circ (du' - X_{H'} \otimes \beta') = (du' - X_{H'} \otimes \beta') \circ j$$

with boundary conditions  $u'(\partial_i E) \subset (\phi^{\tau'_E})^* L_i$  via Liouville pullback. Here, of course,  $H'$  means  $(H^{strict})' + P'$ .

**4.4. Open-closed maps.** The main purpose of the action condition in Definition 4.4 is to pass in a controlled way from holomorphic curves in the fiber to holomorphic curves in the total space. In particular, we will define a weaker version of a nondegenerate stop using an open-closed string map  $\mathcal{OC}: CC_*(\mathcal{B}_\sigma(\sigma)) \rightarrow SC_\sigma^*(M[\sigma])$  which counts holomorphic annuli, as described in [2]. Following Abouzaid, let  $\mathcal{R}_d^1$  be the space of disks with one interior puncture and  $d \geq 1$  boundary punctures, one of which is distinguished. For  $\Sigma \in \mathcal{R}_d^1$ , label the interior puncture by  $\zeta_-$  and the boundary punctures  $\zeta_1$  through  $\zeta_d$ , ordered counterclockwise, with the distinguished puncture labeled  $\zeta_d$ .  $\mathcal{R}_d^1$  has a natural compactification to a manifold with corners  $\overline{\mathcal{R}}_d^1$  whose codimension one faces can be canonically identified with

$$(4.14) \quad \coprod_{\substack{2 \leq k \leq d \\ 1 \leq i \leq k}} \overline{\mathcal{R}}_{d+1-k}^1 \times \overline{\mathcal{R}}^{k+1, i} \quad \amalg \quad \coprod_{\substack{2 \leq k \leq d-1 \\ 1 \leq i \leq d-k}} \overline{\mathcal{R}}_{d+1-k}^1 \times \overline{\mathcal{R}}^{k+1}.$$

Here,  $\overline{\mathcal{R}}^{k+1,i}$  is diffeomorphic to  $\overline{\mathcal{R}}^{k+1}$ , but if  $\Sigma^d \in \overline{\mathcal{R}}^{k+1}$ , then the corresponding point of  $\overline{\mathcal{R}}^{k+1,i}$  is  $\Sigma^d$  with the additional datum that  $\zeta_i \in \Sigma^d$  is distinguished. In other words, it is the space of disks with one negative puncture,  $d$  positive punctures, and such that the  $i$ th puncture is considered special. The first term in (4.14) corresponds then to a collection of punctures which includes  $\zeta_d$  colliding, while the second corresponds to some other collection colliding. In this case, the additional index  $i$  keeps track of where the collision occurred.

A collection of ends for  $\Sigma \in \mathcal{R}_d^1$ , making it into a punctured Riemann surface with boundary, ends, and cylinders, consists of a positive strip-like end  $\epsilon_i$  for each boundary puncture  $\zeta_i$ , along with a negative cylindrical end  $\epsilon_-$  at  $\zeta_-$ . In this case, we ask that  $\epsilon_-$  has a very special form. Specifically, in the holomorphic coordinates on  $\Sigma$  where  $\text{int}(\Sigma) = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$  and  $\zeta_d = 1$ , we require that

$$(4.15) \quad \epsilon_-(s, t) = ae^{2\pi(s+it)} \quad \text{with } a \in \mathbb{R} \text{ positive.}$$

for some positive number  $a \in \mathbb{R}$ . A **universal family of ends** for  $\mathcal{R}_d^1$  consists of a collection of ends on each  $\Sigma \in \mathcal{R}_d^1$  for every  $d$ , such that near the boundary of  $\overline{\mathcal{R}}_d^1$  it agrees up to a rotation of  $\epsilon_-$  with the collection induced by gluing. This rotation correction is unavoidable, since boundary components  $\overline{\mathcal{R}}_{d+1-k}^1 \times \overline{\mathcal{R}}^{k+1,i}$  have the same ends for all  $i$ , so that without rotation at most one could glue to a configuration which satisfies (4.15). However, because we are using an exponential gluing profile the magnitude of the rotation vanishes to infinite order at the boundary, and hence the family of strip-like ends extends smoothly to  $\overline{\mathcal{R}}_d^1$ . One sees as with  $\mathcal{R}^{d+1}$  that universal families of ends for  $\mathcal{R}_d^1$  exist, and we fix one once and for all.

A **universal and conformally consistent** choice of Floer data for  $\mathcal{R}_d^1$  consists of, for all  $d \geq 1$ , a Floer datum  $(\beta, H, P, J, \tau_E)$  for each  $\Sigma \in \mathcal{R}_d^1$  varying smoothly over  $\mathcal{R}_d^1$ , and such that near  $\partial\overline{\mathcal{R}}_d^1$  it agrees to infinite order with the conformal class of not-quite Floer datum determined by gluing. We say not-quite due to the rotation corrections for the strip-like ends, which among other things cause the glued datum to not be a Floer datum in the above sense. Denote by  $\mathcal{K}^{\text{OC}}(M[\sigma])$  the space of universal and conformally consistent choices of Floer data for  $\mathcal{R}_d^1$ .

Given  $\mathbf{K} \subset \mathcal{K}^{\text{OC}}(M[\sigma])$ , we can consider the resulting spaces of holomorphic curves. Given a collection of Lagrangian labels  $L_i$  and asymptotic ends

$$\gamma_i \in \mathcal{X}(L_i, L_{i+1}) \quad \text{and} \quad x_- \in \mathcal{X}(H_t),$$

we are interested in the space

$$\mathcal{R}_d^1(\gamma_d, \dots, \gamma_i; x_-).$$

This consists of all maps  $u: \Sigma \rightarrow \widehat{M[\sigma]}$  for  $\Sigma \in \mathcal{R}_d^1$  satisfying (4.12) with  $u(E_i) \subset (\phi^{\tau_E})^*L_i$ ,  $u(\zeta_i) = (\phi^{\tau_i})^*\gamma_i$ , and  $u(\zeta_-) = (\phi^{\tau_-})^*x_-$ .

**Lemma 4.16.** *There is a dense subset  $\mathcal{K}_{\text{reg}}^{\text{OC}}(M[\sigma]) \subset \mathcal{K}^{\text{OC}}(M[\sigma])$  such that, for every universal choice  $\mathbf{K} \in \mathcal{K}_{\text{reg}}^{\text{OC}}(M[\sigma])$ , the following hold.*

- (1) *For any  $d \geq 1$ , any sequence of Lagrangians  $L_1, \dots, L_d$ , and any collection of chords  $\gamma_i \in \mathcal{X}(L_i, L_{i+1})$  and orbit  $\gamma_- \in \mathcal{X}(H_t)$ ,  $\mathcal{R}_d^1(\gamma_d, \dots, \gamma_i; \gamma_-)$  is a smooth manifold of dimension  $\deg(x_-) - \sum_{i=1}^d \deg(\gamma_i) + d - n - 1$ . Here,  $n$  is half the dimension of  $M$ .*
- (2) *If  $\deg(x_-) - \sum_{i=1}^d \deg(\gamma_i) = n + 1 - d$ , then  $\mathcal{R}_d^1(\gamma_d, \dots, \gamma_i; x_-)$  is compact.*
- (3) *If  $\deg(x_-) - \sum_{i=1}^d \deg(\gamma_i) = n + 2 - d$ , then  $\mathcal{R}_d^1(\gamma_d, \dots, \gamma_i; x_-)$  admits a Gromov compactification as a compact topological 1-manifold with boundary, and its boundary is in natural*

bijection with

$$\begin{aligned}
(4.16) \quad & \coprod_{\substack{2 \leq k \leq d \\ 1 \leq i \leq k \\ \tilde{\gamma} \in \mathcal{X}(L_{d+1-i}, L_{k+1-i})}} \mathcal{R}_{d+1-k}^1(\tilde{\gamma}, \gamma_{d-i}, \dots, \gamma_{1+k-i}; x_-) \times \mathcal{R}^{k+1}(\gamma_{k-i}, \dots, \gamma_{d+1-i}; \tilde{\gamma}) \\
& \coprod_{\tilde{\gamma} \in \mathcal{X}(L_d, L_1)} \mathcal{R}_d^1(\tilde{\gamma}, \gamma_{d-1}, \dots, \gamma_1; x_-) \times \mathcal{R}(\gamma_d; \tilde{\gamma}) \\
& \coprod_{\substack{2 \leq k \leq d-1 \\ 1 \leq i \leq d-k \\ \tilde{\gamma} \in \mathcal{X}(\bar{L}_i, L_{i+k})}} \mathcal{R}_{d+1-k}^1(\gamma_d, \dots, \gamma_{i+k}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_1; x_-) \times \mathcal{R}^{k+1}(\gamma_{i+k-1}, \dots, \gamma_i; \tilde{\gamma}) \\
& \coprod_{\substack{1 \leq i < d \\ \tilde{\gamma} \in \mathcal{X}(\bar{L}_i, L_{i+1})}} \mathcal{R}_d^1(\gamma_d, \dots, \gamma_{i+1}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_1; x_-) \times \mathcal{R}(\gamma_i; \tilde{\gamma}) \\
& \coprod_{\tilde{x} \in \mathcal{X}(H_t)} \mathcal{Q}(\tilde{x}; x_-) \times \mathcal{R}_d^1(\gamma_d, \dots, \gamma_1; \tilde{x}).
\end{aligned}$$

*Remark on proof.* To address the compactness, recall from Definition 4.15 that the perturbing Hamiltonian has no effect outside the 2-shifts of the cylinders. On the other hand,  $J$  is of rescaled contact type in the 2-shifts of the cylinders. Thus, we can separate the compactness problem into the 2-shifts of the cylinders and their complement.

In the first case, the  $C^0$  estimates in [19] give a bound on how far elements  $u \in \mathcal{R}_d^1(\gamma_d, \dots, \gamma_i; \gamma_-)$  can protrude into the symplectization. Everywhere else, Lemma A.8 applies. Together, these show that the image of any  $u$  is constrained to lie in a compact set depending only on  $\mathbf{K}_\Delta$  and the ends  $\gamma_i$  and  $\gamma_-$ .  $\square$

Define  $\mathcal{OC}: CC_*(\mathcal{B}_\sigma(\sigma)) \rightarrow SC^{*+n}(M[\sigma])$  by

$$\mathcal{OC}(\gamma_d \otimes \dots \otimes \gamma_1) = \sum_{\substack{x \in \mathcal{X}(H_t) \\ \deg(x) = \sum_{i=1}^d \deg(\gamma_i) + n + 1 - d}} \# \mathcal{R}_d^1(\gamma_d, \dots, \gamma_i; x) \cdot x.$$

The boundary strata in (4.16) tell us that  $\mathcal{OC}$  is a chain map. Further, arguing as in Lemma 4.12 gives

**Lemma 4.17.** *The image of  $\mathcal{OC}$  lies in  $SC_\sigma^*(M[\sigma])$ .*  $\square$

**Definition 4.18.** A stop  $\sigma \in \sigma$  is **weakly nondegenerate** if, for some choice of Floer data for symplectic cohomology, the Fukaya category and the open-closed map, there is a closed Hochschild chain  $y \in CC_{1-n}(\mathcal{B}_\sigma(\sigma))$  such that  $\mathcal{OC}(y) = f_\sigma$ , where  $f_\sigma \in SC_\sigma^1(M[\sigma])$  is a **saddle unit** of  $\sigma$  as described below.

Morally speaking, a saddle unit is any cocycle which lives in the central fiber of  $\Sigma F \subset M[\sigma]$  and represents the unit of  $SH^*(F)$  when thought of as a chain in  $SC^0(F)$ . The drop in degree from 1 to 0 comes from the fact that the central fiber lives at a saddle point of the Liouville vector field for  $\mathbb{C}_1$ , which translates to an index 1 Morse critical point for nice choices of compatible Hamiltonian. However, such a cocycle is often exact in  $SC_\sigma^*(M[\sigma])$ , so the careful definition of  $f_\sigma$  instead involves a count of holomorphic caps.

Concretely, let  $\Sigma$  be  $\mathbb{C}$  equipped with the negative cylindrical end  $\epsilon^f$  asymptotic to  $\infty$  given by

$$\epsilon^f(s, t) = e^{-2\pi(s+it)}$$

Let  $\mathcal{K}^\mathbb{C}(M[\sigma])$  denote the space of Floer data on  $\Sigma$ . Given a Floer datum  $K^f \in \mathcal{K}^\mathbb{C}(M[\sigma])$  and an orbit  $x \in \mathcal{X}(H_t)$ , we are interested in the resulting moduli space  $\mathcal{C}(x)$ . This is the space of

all maps  $u: \Sigma \rightarrow \widehat{M[\sigma]}$  satisfying (4.12) and

$$\lim_{s \rightarrow -\infty} u(\epsilon^f(s, t)) = (\phi^\tau)^* x(t),$$

where  $\tau$  is the conformal factor  $K^f$  assigns to  $\infty \in \bar{\Sigma}$ , and for which

$$u(0) \in Y_\sigma,$$

where  $Y_\sigma \subset \hat{M}[\sigma]$  is the hypersurface which comes from  $\hat{F} \times i\mathbb{R} \subset \Sigma F$ . The last condition is the interesting one. Indeed, that is the only place where the stop  $\sigma$  comes into the definition of  $f_\sigma$ , and without it we would just obtain the unit of symplectic cohomology.

**Lemma 4.19.** *There is a comeager subset  $\mathcal{K}_{reg}^{\mathbb{C}}(M[\sigma]) \subset \mathcal{K}^{\mathbb{C}}(M[\sigma])$  such that, for any  $K^f \in \mathcal{K}_{reg}^{\mathbb{C}}(M[\sigma])$ , the following hold.*

- (1) *For all  $x \in \mathcal{X}(H_t)$ ,  $\mathcal{C}(x)$  is a smooth manifold of dimension  $\deg(x) - 1$ .*
- (2) *If  $\deg(x) = 1$ , then  $\mathcal{C}(x)$  is compact.*
- (3) *If  $\deg(x) = 2$ , then  $\mathcal{C}(x)$  has a Gromov compactification  $\bar{\mathcal{C}}(x)$  which is a compact topological 1-manifold with boundary, and there is a canonical identification*

$$\partial \bar{\mathcal{C}}(x) = \coprod_{\tilde{x} \in \mathcal{X}(H_t)} \mathcal{Q}(\tilde{x}; x) \times \mathcal{C}(\tilde{x}).$$

*In this case,  $\tilde{x}$  necessarily has degree 1.*

□

**Definition 4.20.** A **saddle unit** of  $\sigma$  is any chain

$$f_\sigma = \sum_{\substack{x \in \mathcal{X}(H_t) \\ \deg(x)=1}} \# \mathcal{C}(x) \cdot x$$

obtained from a Floer datum  $K^f \in \mathcal{K}_{reg}^{\mathbb{C}}(M[\sigma])$ . It follows from Lemma 4.19 and positivity of intersections that such a chain is in fact a closed element of  $SC_\sigma^1(M[\sigma])$ .

The rest of Section 4 is devoted to proving

**Proposition 4.21.** *Every strongly nondegenerate stop is weakly nondegenerate.*

**4.5. Stabilizations.** The proof of Proposition 4.21 amounts to a very careful choice of Floer data. To do this, we first consider the case of a stabilization, and there it will be helpful to work with a special Liouville form on  $\mathbb{C}_1$ . For this, let  $\lambda_{\mathbb{C}_1}^0$  be the Liouville form on  $\mathbb{C}_1$  constructed in Example 2.10. Near the origin, it is given by

$$\lambda_{\mathbb{C}_1}^0 = \left(-1 + \frac{1}{2}\epsilon + x^2\right) xdy - \left(1 + \frac{1}{2}\epsilon + y^2\right) ydx.$$

Let  $\kappa$  be the cutoff function of Example 2.10, so that  $\kappa$  is radially invariant, equals 1 when  $|z| < \frac{1}{4}$ , and equals 0 when  $|z| > \frac{1}{2}$ . The Liouville form  $\lambda_{\mathbb{C}_1} = \lambda_{\mathbb{C}_1}^0 + d(xy\kappa(|z|))$  given near zero by

$$\lambda_{\mathbb{C}_1} = \left(\frac{1}{2}\epsilon + x^2\right) xdy - \left(\frac{1}{2}\epsilon + y^2\right) ydx$$

is still invariant under the involution  $z \mapsto -z$ , and  $i\mathbb{R}$  is still invariant under its Liouville vector field  $\hat{Z}_{\mathbb{C}_1}$ . Moreover, it has two very nice properties which  $\lambda_{\mathbb{C}_1}^0$  lacks. First, its Liouville vector field agrees to second order at the origin with  $Z_{std} = \frac{1}{2}(x\partial_x + y\partial_y)$ . Second, it receives a non-proper Liouville embedding

$$I: (\mathbb{C}, \lambda_{std}) \rightarrow (\mathbb{C}_1, \lambda_{\mathbb{C}_1})$$

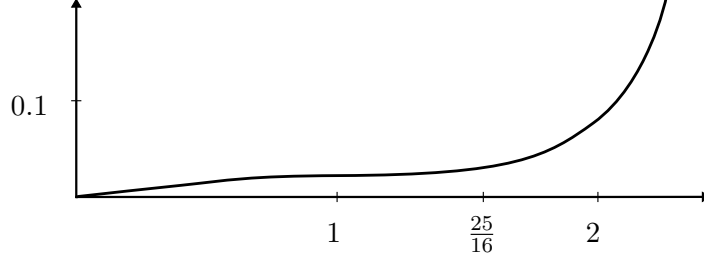


FIGURE 6.

which is  $\mathbb{Z}/2$ -equivariant, strictly preserves the Liouville forms outside  $I^{-1}(D_{\frac{1}{4}})$ , sends  $\mathbb{R}$  into  $\mathbb{R}$  and  $i\mathbb{R}$  onto  $i\mathbb{R}$ , and whose image avoids the images of the stops.

With this, let  $F$  be a strongly nondegenerate Liouville domain, and consider  $N = \Sigma F$  with Liouville form  $\lambda_N = \lambda_F + \lambda_{\mathbb{C}_1}$ . Denote by  $\Phi: \hat{N} \rightarrow \hat{N}$  the involution which is identity in the  $\hat{F}$  component and rotation by  $\pi$  in the  $\mathbb{C}_1$  component. In particular,  $\Phi$  exchanges the two stops in  $N$ . For  $\varepsilon > 0$  and an object  $L \in \mathcal{W}^\varepsilon(F)$ , we can build  $\Phi$ -invariant Lagrangians in  $N$  as follows.

Let  $f$  be the unique compactly supported primitive of  $\hat{\lambda}_F|_L$ , which exists by the conditions on Lagrangians in  $\mathcal{W}^\varepsilon(F)$ . Choose  $\tau: \mathbb{R} \rightarrow \mathbb{R}$  to be a smooth, *even* function which vanishes in a neighborhood of zero and, outside a compact set, is of the form

$$\tau(y) = 2 \log y + \log B$$

for some constant  $B > 0$ . Write  $\phi^\tau$  for the time  $\tau$  flow of  $\hat{Z}_N$ . Then the map

$$(p, y) \mapsto \left( \phi^{\tau(y)}(p), \left( -f(p) \frac{d}{dy} e^{\tau(y)}, y \right) \right)$$

is a Lagrangian embedding  $\Sigma_L^\tau: L \times \mathbb{R} \hookrightarrow (\hat{F} \times \mathbb{C}, \hat{\lambda}_F + \hat{\lambda}_{\mathbb{C}, \text{std}})$  whose image is  $\Phi$ -invariant globally and  $\hat{Z}_{F \times \mathbb{C}}$ -invariant outside a compact set. Note that, for  $\tau$  not too wild on the compact part of its domain, we have

$$\left| \frac{d}{dy} e^{\tau(y)} \right| < 2B|y|,$$

which together with the variance condition on  $f$  implies that

$$(4.17) \quad |x| < 2\varepsilon B|y|.$$

Let  $C > 0$  be such that  $|I(z)| > 1$  for  $z \in i\mathbb{R}$  with  $|z| > C$ , and choose once and for all such a  $\tau$  which additionally vanishes on  $[-C, C]$ . Then  $(Id, I) \circ \Sigma_L^\tau: L \times \mathbb{R} \hookrightarrow N$  is a Lagrangian embedding which is split wherever  $(Id, I)$  fails to preserve the Liouville form. This implies that its image is a conical Lagrangian in  $N$ , and we denote it by  $\Sigma L$ .

In the remainder of this section, we construct Hamiltonians on  $N$  which are simultaneously well-adapted to the splitting  $F \times \mathbb{C}_1$  and close to being normalizing Hamiltonians. For this, let  $H^s$  be a compatible Hamiltonian on  $F$  satisfying  $dH^s(\hat{Z}_F) \leq 2H^s$  globally and  $H^s < 0.1$  whenever  $dH^s(\hat{Z}_F) \neq 2H^s$ . Let  $f_0: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be a smooth, increasing function which satisfies

- (1) There is some  $a > 0$  such that  $f_0(x) = ax$  for  $x$  close to zero,
- (2)  $f'_0(x) \leq a$  for  $x \leq \frac{25}{16}$ ,
- (3)  $f'_0(1) = 0$ ,
- (4)  $f_0(x) = x^2$  for  $x$  sufficiently large,
- (5)  $f_0(2) < 0.1$ , and
- (6)  $f'_0(x) \geq 2x$  for  $x \geq 2$ .

See Figure 6.

Given  $\alpha > 0$ , write  $H^\alpha: N \rightarrow \mathbb{R}$  for  $\frac{1}{\alpha}$  times the compatible Hamiltonian given by Equation (2.6) with  $g = \alpha^2 H^s$  and  $f(z) = \alpha f_0(|z^2 - 1|^2)$ . Setting  $K = (\alpha H^s)^{-1}((-\infty, 2]) \subset \hat{F}$ , we have

$$(4.18) \quad H^\alpha|_{K \times D_{0.9}}(p, z) = \alpha H^s(p) + f_0(|z^2 - 1|^2) \quad \text{for all } \alpha \text{ sufficiently small.}$$

In this case,  $X_{H^\alpha}$  has the crucial property

$$(4.19a) \quad dx(X_{H^\alpha}) \text{ has the opposite sign as } y \text{ and}$$

$$(4.19b) \quad dy(X_{H^\alpha}) \text{ has the opposite sign as } x,$$

where  $z = x + iy$  is the coordinate on  $D_{0.9}$ .

Let  $\kappa: \mathbb{R} \rightarrow [0, 1]$  be a nondecreasing cutoff function with  $\kappa(x) = 0$  for  $x \leq 1$  and  $\kappa(x) = 1$  for  $x \geq 2$ . Such a  $\kappa$  satisfies  $\kappa(f_0(|z^2 - 1|^2)) = 0$  for  $z \in D_{\frac{1}{2}}$  and  $\kappa(f_0(|z^2 - 1|^2)) = 1$  for  $z \in D_{0.9}$  with  $|\Im(z)| > 0.8$ . Define  $H^N: N \rightarrow \mathbb{R}$  by

$$(4.20) \quad H^N = c H^\alpha \kappa(H^\alpha).$$

where  $c > 0$  is a small constant which ensures that  $H^N$  is  $C^2$ -small on the interior of  $N$ . This is not quite a normalizing Hamiltonian on  $N$ , but it has some very nice properties.

**Lemma 4.22.** *The following hold.*

- (1)  $H^N$  is  $\Phi$ -invariant on  $K \times D_{0.9}$ , and there it satisfies (4.19) whenever  $dH^N \neq 0$ .
- (2)  $H^N \geq 2c$  on  $K \times (D_{0.9} \cap \{|\Im(z)| > 0.8\})$  and on  $\partial K \times D_{0.9}$ .
- (3)  $H^N = 0$  in a neighborhood of  $\{(p, z) \in \hat{N} \mid \hat{Z}_N(p, z) = 0\}$ .
- (4) The restriction  $H^N|_{\hat{F} \times \{0\}}$  is a normalizing Hamiltonian on  $F$ .
- (5) Along  $K \times \{0\}$ , the vector field  $X_{H^N}$  vanishes in the  $\mathbb{C}_1$ -directions to at least third order.
- (6) For sufficiently small  $a$  in the definition of  $f_0$  and sufficiently small  $\alpha$  in the definition of  $H^\alpha$ , the following holds. For all  $\tau \geq 1$ ,  $H_\tau^N \geq \tau^{3/4} H^N$  everywhere.

*Proof.* The first five are immediate consequences of the construction. The sixth follows from a sequence of straightforward but tedious calculations, which we now perform.

The conclusion is equivalent to the statement that

$$(4.21) \quad dH^N(\hat{Z}_N) \geq \frac{7}{4} H^N$$

everywhere. In this case, we have

$$\begin{aligned} dH^N(\hat{Z}_N) &= c(\kappa(H^\alpha) + \kappa'(H^\alpha)) dH^\alpha(\hat{Z}_M) \\ &= \left(1 + \frac{\kappa'(H^\alpha)}{\kappa(H^\alpha)}\right) \cdot d\log(H^\alpha)(\hat{Z}_M) \cdot H^N, \end{aligned}$$

so it suffices to prove that  $d\log H^\alpha(\hat{Z}_M) \geq \frac{7}{4}$ , or equivalently that  $H^\alpha$  satisfies equation (4.21), whenever  $H^\alpha \geq 1$ .

We begin with the case that  $H^\alpha$  is split, which occurs when  $f(z)$  and  $g(p)$  are both smaller than 1. In this case, we have  $dH^\alpha \geq 2H^\alpha$  except where at least one of  $f_0(|z^2 - 1|^2)$  and  $H^s(p)$  is smaller than 0.1. The condition  $H^\alpha \geq 1$  implies that this can only happen when at least one of  $f_0(|z^2 - 1|^2)$  and  $\alpha H^s(p)$  is at least 0.9.



If  $f_0(|z^2 - 1|^2) > 0.9$  and  $\alpha H^s(p) < H^s(p) < 0.1$ , then we compute

$$\begin{aligned}
dH^\alpha(\hat{Z}_N) &= d(f_0(|z^2 - 1|^2))(\hat{Z}_{\mathbb{C}_1}) + \alpha dH^s(\hat{Z}_F) \\
&\geq 2|z^2 - 1|^2 \cdot d(|z^2 - 1|^2)(\hat{Z}_{\mathbb{C}_1}) + \alpha dH^s(\hat{Z}_F) \\
&= 2|z^2 - 1|^4 + \alpha dH^s(\hat{Z}_F) \\
&\geq 2f_0(|z^2 - 1|^2) + \alpha dH^s(\hat{Z}_F) \\
&\geq 1.8H^\alpha + \alpha dH^s(\hat{Z}_F) \\
&= \left(1.8 + \alpha \frac{dH^s(\hat{Z}_F)}{H^\alpha}\right) \cdot H^\alpha.
\end{aligned}$$

Now  $dH^s(\hat{Z}_F)$  is positive outside a compact set, so it is bounded below. This means we can take  $\alpha$  small enough that  $\alpha dH^s(\hat{Z}_F) > -0.05$ . Because  $H^\alpha \geq 1$  by assumption, this gives

$$dH^\alpha(\hat{Z}_N) \geq 1.75H^\alpha$$

as desired.

If instead  $\alpha H^s(p) > 0.9$  and  $f_0(|z^2 - 1|^2) < 0.1$ , the proof is similar, except we use extra information about the Liouville form to accomplish the same thing by reducing only  $a$ . In this case, the same calculation gives

$$dH^\alpha(\hat{Z}_N) \geq \left(1.8 + \frac{d(f_0(|z^2 - 1|^2))(\hat{Z}_{\mathbb{C}_1})}{H^\alpha}\right) \cdot H^\alpha.$$

Now recall that  $\hat{\lambda}_{\mathbb{C}_1} = u^* \hat{\lambda}_{\mathbb{C}, \text{std}}$  outside  $D_{\frac{1}{2}}$ , where  $u(z) = z^2 - 1$ . This implies that

$$d(f_0(|z^2 - 1|^2))(\hat{Z}_{\mathbb{C}_1}) = f'_0(|z^2 - 1|^2) \cdot |z^2 - 1|^2 \geq 0$$

for  $z \notin D_{\frac{1}{2}}$ , which gives

$$dH^\alpha(\hat{Z}_N) \geq 1.8H^\alpha.$$

On the other hand, for  $z \in D_{\frac{1}{2}}$  we have  $|z^2 - 1|^2 \leq \frac{25}{16}$ , so that

$$d(f_0(|z^2 - 1|^2))(\hat{Z}_{\mathbb{C}_1}) \geq \min \left\{0, ad(|z^2 - 1|^2)(\hat{Z}_{\mathbb{C}_1})\right\}.$$

The term  $d(|z^2 - 1|^2)(\hat{Z}_{\mathbb{C}_1})$  is again bounded below, so it is again bigger than  $-0.05$  for sufficiently small  $a$ .

When  $H^\alpha$  is not split but not quadratic, the situation is similar to the above, except now we may need to shrink  $\alpha$  further to manage the new terms which come from differentiating (2.6). Writing this out, we have

$$H^\alpha = \frac{1}{\alpha} \left[ e^{2h(f(z))} g(\phi_F(-h(f(z)), p)) + e^{2h(g(p))} f(\phi_{\mathbb{C}_1}(-h(g(p)), z)) \right],$$

where  $h: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is a nondecreasing function which equals 0 for  $x \leq 1$  and  $\frac{1}{2} \log x$  for  $x \geq 2$ .  $H^\alpha$  is quadratic or split unless all of the following hold.

- (1) Both  $f(z)$  and  $g(p)$  are smaller than 2.
- (2) At least one of  $f(z)$  and  $g(p)$  is at least 1.
- (3) Either  $\frac{g(p)}{f(z)} < 0.1\alpha^2$  or  $\frac{f(z)}{g(p)} < x_0\alpha$ , where  $x_0$  is such that  $f_0(x) = x_0^2$  for  $x \geq x_0$ .

By shrinking  $\alpha$  so that  $\alpha x_0 \ll 1$ , the above simplifies into two disjoint regions:

- (1)  $f(z) \in [1, 2]$  and  $H^s(p) < 0.2$ , or
- (2)  $g(z) \in [1, 2]$  and  $|z_2 - 1|^2 < 2x_0$ .

The two cases behave essentially identically, so we consider only the first. In this case, the second term

$$\frac{1}{\alpha} e^{2h(g(p))} f(\phi_{C_1}(-h(g(p)), z)) = f_0(|z^2 - 1|^2)$$

is strictly quadratic and  $O(\frac{1}{\alpha})$ . For the first term, we calculate

$$\begin{aligned} & d \left[ e^{2h(f)} g(\phi_F(-h(f), p)) \right] (\hat{Z}_N) \\ &= 2e^{2h(f)} h'(f) [4g(\phi_F(-h(f), p)) - dg(\phi_F(-h(f), p))(Z_F)] \cdot f \\ &+ e^{2h(f)} dg(\phi_F(-h(f), p))(Z_F). \end{aligned}$$

The coefficients  $e^{2h(f)}$  and  $f$  are each bounded by 2, while  $h'(f)$  has an upper bound depending only on  $h$ . This means that both

$$\frac{1}{\alpha} e^{2h(f)} g(\phi_F(-h(f), p))$$

and

$$d \left[ \frac{1}{\alpha} e^{2h(f)} g(\phi_F(-h(f), p)) \right] (\hat{Z}_N)$$

are  $O(\alpha)$ . This gives

$$dH^\alpha(\hat{Z}_N) = (2 + O(\alpha^2))H^\alpha$$

and completes the proof.  $\square$

For the rest of Section 4, fix  $H^\alpha$  and  $H^N$  satisfying the conclusion of Lemma 4.22.

**4.6. Collapsing the energy.** Moving to the global situation, let  $M$  be a pumpkin domain of the form  $M'[\sigma]$ , where  $\sigma$  is strongly nondegenerate with fiber  $F$ . Choose the Liouville form  $\hat{\lambda}_M$  on  $M$  to strictly agree with  $\hat{\lambda}_N$  on the gluing region. We will extend the Hamiltonian  $H^N$  to a Hamiltonian  $H^M$  and define a family of spaces  $\mathcal{H}_\varepsilon(M)$  as if  $H^M$  were a normalizing Hamiltonian. With this, we will be able to control the open-closed maps by degenerating  $\varepsilon$ .

To begin, assume  $\partial N \subset \hat{N}$  is chosen to be large enough to easily contain both stops, e.g.  $\partial N = (H^N)^{-1}(100)$ . Let  $U \subset \partial N$  be the intersection

$$\partial N \cap \sigma_0(\hat{F} \times \{\Re(z) > 0\}),$$

and let  $P_0 \subset U$  be the intersection

$$\partial N \cap \sigma_0(\hat{F} \times \{\Re(z) > \rho\}),$$

where  $\rho$  is the width of the stop  $\sigma$  on  $M'$ . Note that  $\partial N \setminus P_0$  embeds canonically into  $M$ , and that this embedding strictly intertwines the contact 1-forms. Let  $P \subset U$  be a compact subset containing  $P_0$  such that the positive flow of  $\hat{Z}_M$  is proper on  $\partial N \setminus P$ .

Pick a smooth function  $g: \partial N \rightarrow [0, 1]$  which equals 0 on a neighborhood  $P$  and 1 outside  $U$ . This gives rise to a function  $H_g^N = H^N \tilde{g}$ , where  $\tilde{g}$  is the  $\hat{Z}_N$ -invariant extension of  $g$ . By (3) in Lemma 4.22,  $H_g^N$  is a smooth function on  $\hat{N}$ . From the construction of  $P$ , we see that it extends by 0 to a smooth function on  $\hat{M}$ , which we continue to call  $H_g^N$ .

Analogously, pick a normalizing Hamiltonian  $H^{M'}$  on  $M'$ , and cut it off with a function  $g'$  so that it extends to a smooth function  $H_{g'}^{M'}$  on  $M$ . By choosing the analog of  $P$  appropriately, we may assume that  $H_{g'}^{M'}$  vanishes on  $K \times D_{0.9}$ . With this, define

$$(4.22) \quad H^M := H_g^N + H_{g'}^{M'}.$$

It has the following properties.

- (1)  $H^M$  is proper, nonnegative, and satisfies  $dH^M(\hat{Z}_M) = 2H^M$  outside a compact set.

- (2) For all  $\tau \geq 1$ ,  $H_\tau^M \geq \tau^{3/4} H^M$  everywhere.
- (3)  $H^M = 0$  in a neighborhood of  $\{p \in \hat{M} \mid \hat{Z}_M(p) = 0\}$ .
- (4)  $H^M$  agrees with  $H^N$  on  $K \times D_{0.9}$  and  $\hat{F} \times \{0\}$ . In particular,  $H^M|_{\hat{F} \times \{0\}}$  is a normalizing Hamiltonian on  $F$ .
- (5) After possibly replacing all Hamiltonians with  $\epsilon$ -multiples of themselves, all nonconstant time 1 orbits of  $X_{H^M}$  live in the conical region where  $dH^M(\hat{Z}_M) = 2H^M$ , and moreover none of these occurs in  $K \times \{0\}$ .

Although it vanishes sometimes,  $H^M$  is close to being a compatible Hamiltonian, and so we obtain spaces  $\mathcal{J}(M, H)$  and  $\mathcal{J}_{resc}^{S^1}(M, H)$  of adapted and rescaled almost complex structures. Choose almost complex structures  $J^M \in \mathcal{J}(M, H)$  and  $J_{resc}^M \in \mathcal{J}_{resc}^{S^1}(M, H)$  which on  $K \times D_{0.9}$  are split and take the form  $(J^F, J_{std})$ . We will treat  $(H^M, J^M)$  and  $(H^M, J_{resc}^M)$  as normalizing Floer data for Lagrangian Floer cohomology and symplectic cohomology, respectively.

In that spirit, choose universal and conformally consistent families of Floer data  $\mathbf{K}^\mu$  for the  $A_\infty$  structure and  $\mathbf{K}^{OC}$  for the open-closed maps, along with a Floer datum  $K^C$  for the saddle unit, such that the following hold.

- (1) All Hamiltonians agree up to rescaling with  $H^M$  and satisfy

$$d^\Sigma H \wedge \beta \leq 0$$

globally.

- (2) After the same rescaling as with the Hamiltonians, all almost complex structures are split on  $K \times D_{0.9}$ , and there they take the form  $(J^F, J_{std})$  for some potentially domain-dependent  $J^F$ .
- (3) For every domain  $\Sigma$  with nonempty boundary, each boundary component  $E$  of  $\Sigma$  contains disjoint open intervals  $E^+$  and  $E^-$  such that  $\tau_E$  is nonincreasing outside  $E^+$  and nondecreasing outside  $E^-$ .
- (4) The rescaling factors are all at least 1 and satisfy

$$(4.23) \quad e^{1-d} \tau_0 < \tau_E < \tau_0$$

for the  $A_\infty$  disks  $\Sigma^{d+1} \in \mathcal{R}^{d+1}$ ,

$$(4.24) \quad e^{1-d} \tau_- < \tau_E < \tau_-$$

for the open-closed annuli  $\Sigma_d^1 \in \mathcal{R}_d^1$ , and

$$\tau = 1$$

for the saddle unit.

- (5) The 1-form  $\beta^C$  for the saddle unit satisfies  $\beta^C = f(r)d\theta$ , where  $f$  is a nonincreasing function which equals 0 near zero and  $\frac{1}{2\pi}$  outside a compact set.

Conditions (4.23) and (4.24) are more difficult than before, but they can still be achieved. In this case the minimal coefficient is  $(2^{4/3})^{d-1}$ , but  $e$  is bigger than  $2^{4/3}$ , so we're fine. We will use these Floer data as normalizations rather than to define holomorphic curve counts directly, so the complete lack of nondegeneracy of critical points and regularity of moduli spaces is not a problem.

From here, choose for all  $\mathbf{m} \in \mathbb{N}$  a nondegenerate compatible Hamiltonian and regular almost complex structure for each pair of Lagrangians, along with a regular Floer datum  $(H_t^{\mathbf{m}}, J_t^{\mathbf{m}})$  for the symplectic cochain complex. Extend these to universal and conformally consistent, regular families of Floer data  $\mathbf{K}^{\mu, \mathbf{m}}$  for the  $A_\infty$  operations,  $\mathbf{K}^{OC, \mathbf{m}}$  for the open-closed maps (compatibly with  $\mathbf{K}^{\mu, \mathbf{m}}$ ), and  $K^{C, \mathbf{m}}$  for the saddle unit, in such a way that the following hold.

- (1)  $\mathbf{K}^{\bullet, \mathbf{m}}$  is  $\frac{1}{2\mathbf{m}}$ -close to  $\mathbf{K}^\bullet$  in  $C^{\mathbf{m}}$ . Here, we use a notion of closeness which allows small changes at infinity, as in (4.2).

- (2) For fixed inputs, the resulting perturbed holomorphic disks satisfy a uniform-in- $\mathbf{m}$  maximum principle. As with the Floer data appearing in the proof of Proposition 3.32, this can be achieved by only allowing perturbations near infinity where  $d^\Sigma H$  is supported in the set  $\{d\beta \leq c \cdot d\text{vol}\}$  for some fixed negative number  $c$ .
- (3) For every domain  $\Sigma$  with nonempty boundary, each boundary component  $E$  of  $\Sigma$  contains disjoint open intervals  $E^+$  and  $E^-$  such that  $\tau_E$  is nonincreasing outside  $E^+$  and nondecreasing outside  $E^-$ . Additionally, the  $\tau_E$  all satisfy (4.23) or (4.24) up to an additive error of  $\frac{1}{\mathbf{m}}$ .
- (4) For all

$$\Sigma \in \bigcup_{d=1}^k \overline{\mathcal{R}}^{d+1} \cup \bigcup_{d=1}^k \overline{\mathcal{R}}_d^1$$

where  $k$  is the Hochschild length of  $F$ , there is a function  $\tau^\mathbf{m}: \Sigma \rightarrow [1, \infty)$  such that

- (a)  $\tau^\mathbf{m}$  agrees on each strip or cylinder with the rescaling factor associated to that strip or cylinder, and elsewhere it extends the function  $\tau_E$ .
- (b) The un-rescaled family of almost complex structures  $J_u^{\bullet, \mathbf{m}}$ , where  $u = \frac{1}{\tau^\mathbf{m}}$ , is split on  $K \times D_{0.9}$  and agrees with  $J_{std}$  in the second component.
- (c) The un-rescaled family of Hamiltonians  $H_u^{\bullet, \mathbf{m}}$  is  $\Phi$ -invariant on  $K \times D_{0.9}$ , and there it satisfies (4.19a) whenever  $|y| > 0.1$  and (4.19b) whenever  $|x| > 0.1$ .
- (d) All critical points of  $H_u^{\bullet, \mathbf{m}}|_{Y_\sigma}$  lie in the central fiber  $K \times \{0\}$ . Moreover, along the central fiber  $H_u^{\bullet, \mathbf{m}}$  agrees to second order with

$$H_u^{\bullet, \mathbf{m}}|_{K \times \{0\}} + a(y^2 - x^2)$$

for some  $a > 0$ .

- (5) For  $\Sigma = \mathbb{C}$ , the above holds with  $u \equiv 1$ . Further, the 1-form  $\beta^{\mathbb{C}, \mathbf{m}}$  satisfies  $\beta^{\mathbb{C}, \mathbf{m}} = f_\mathbf{m}(r)d\theta$ , where  $f_\mathbf{m}$  is a nonincreasing function which equals 0 near zero and  $\frac{-1}{2\pi}$  outside a compact set.
- (6) The interior action leaks satisfy the bounds

$$\int_{\Sigma^{d+1}} \max_{\hat{M}}(d^\Sigma H^{\mu, \mathbf{m}} \wedge \beta^{\mu, \mathbf{m}}) < \frac{2de^{d-1}}{\mathbf{m}\tau_0}$$

for  $\Sigma^{d+1} \in \mathcal{R}^{d+1}$ ,

$$\int_{\Sigma_d^1} \max_{\hat{M}}(d^\Sigma H^{\mathcal{OC}, \mathbf{m}} \wedge \beta^{\mathcal{OC}, \mathbf{m}}) < \frac{2de^{d-1} + 1}{\mathbf{m}\tau_0}$$

for  $\Sigma_d^1 \in \mathcal{R}_d^1$ , and

$$\int_{\Sigma} \max_{\hat{M}}(d^\Sigma H^{\mathbb{C}, \mathbf{m}} \wedge \beta^{\mathbb{C}, \mathbf{m}}) < \frac{1}{\mathbf{m}}.$$

for  $\Sigma = \mathbb{C}$ .

*Remark 4.23.* For perturbed holomorphic curves living entirely in  $K \times D_{0.9}$ , one needs access to  $\mathbb{Z}/2$  transversality statements. This is straightforward in the stable case outside the central fiber, and in the unstable case one can argue as in [24]. For  $A_\infty$  disks lying entirely in the central fiber, the situation is essentially identical to that in Section 14 of [33], where Seidel's topological argument guarantees transversality. For domains with negative cylindrical ends, however, the topological argument fails. We address the regularity of the moduli spaces of such curves in section 4.7, where it will turn out that our argument only guarantees transversality when  $\mathbf{m}$  is large and the full family of domains is compact. This latter condition is why we only ask  $H$  to be  $\Phi$ -invariant up to  $d = k$ .

We are now prepared to begin proving Proposition 4.21. To do so, we will want to show that for large  $\mathbf{m}$ , essentially all relevant holomorphic curves live in the central fiber. In principle one could argue this as a strict Morse degeneration, but for our purposes the following lemma suffices.

**Lemma 4.24.** *Let  $\Sigma$  be a finite strip or cylinder  $[a, b] \times C$ , where  $C$  is either  $[0, 1]$  or  $S^1$ . If  $\Sigma$  is a strip, equip it with Lagrangian labels  $\Sigma L_0$  along  $[a, b] \times \{0\}$  and  $\Sigma L_1$  along  $[a, b] \times \{1\}$ , where  $L_0$  and  $L_1$  are objects of  $\mathcal{W}^\varepsilon(F)$  for some  $\varepsilon > 0$ . Equip  $\Sigma$  with a Floer datum  $(H^\mathbf{m}, J^\mathbf{m})$  coming from the above choices for some finite  $\mathbf{m}$ , and suppose  $u: \Sigma \rightarrow \hat{M}$  is a perturbed holomorphic map whose image lies in  $K \times D_{0.9}$ . Then*

$$(4.25) \quad \pm (y \circ u) > 0.1 \text{ globally} \implies \pm \left( \int_C y(u(a, t)) dt - \int_C y(u(b, t)) dt \right) > 0,$$

$(x, y)$  is the coordinate function on  $D_{0.9}$ . If  $C = S^1$ , then we also have

$$(4.26) \quad \pm (x \circ u) > 0.1 \text{ globally} \implies \mp \left( \int_C x(u(a, t)) dt - \int_C x(u(b, t)) dt \right) > 0.$$

*Proof.* The proof is identical in all four cases, so we restrict to the case  $y \circ u > 0.1$ . Then

$$\begin{aligned} \left( \int_C y(u(a, t)) dt - \int_C y(u(b, t)) dt \right) &= \int_\Sigma -d(y \circ u)(\partial_s) ds \wedge dt \\ &= \int_\Sigma -dy \circ du \wedge dt \\ &= \int_\Sigma -dy \circ (du - X_{H^\mathbf{m}} dt) \wedge dt \\ &= \int_\Sigma dy \circ J^\mathbf{m} \circ (du - X_{H^\mathbf{m}} dt) \circ j \wedge dt \\ \text{(using that } J^\mathbf{m} \text{ is split)} &= \int_\Sigma -dx \circ (du - X_{H^\mathbf{m}} dt) \circ j \wedge dt \\ &= \int_\Sigma -d(x \circ u)(\partial_t) ds \wedge dt + \int_\Sigma dx(X_{H^\mathbf{m}}) ds \wedge dt \\ \text{(using that } H^\mathbf{m} \text{ satisfies (4.19a))} &< \int_\Sigma -d(x \circ u)(\partial_t) ds \wedge dt \\ &= - \int_a^b \int_C \frac{\partial(x \circ u(s, \cdot))}{\partial t} dt ds = - \int_a^b 0 ds = 0. \end{aligned}$$

In the last step, when  $C = [0, 1]$ , it is crucial that the Lagrangian boundary conditions lie along  $\{x = 0\}$ . This is where we use the vanishing condition on  $\tau$  in the construction of  $\Sigma L$ .  $\square$

**Lemma 4.25.** *Let*

$$\mathcal{D} \subset (\mathcal{R}^{*+1} \cup \mathcal{R}_*^1 \cup \{\mathbb{C}\} \cup \{Z\} \cup \{\mathbb{R} \times S^1\})$$

*be a compact space of domains, where*

$$\mathcal{R}^{*+1} := \bigcup_{d=1}^{\infty} \overline{\mathcal{R}}^{d+1} \quad \text{and} \quad \mathcal{R}_*^1 := \bigcup_{d=1}^{\infty} \overline{\mathcal{R}}_d^1.$$

*Then there is some  $\mathbf{m}_0 \in \mathbb{N}$  and  $E_0 > 0$  such that any holomorphic curve  $u$  with domain in  $\mathcal{D}$ , boundary conditions on Lagrangians of the form  $\Sigma L$ , Floer datum  $K^{\bullet, \mathbf{m}}$  with  $\mathbf{m} > \mathbf{m}_0$ , geometric energy and output action below  $E_0$ , and all inputs in the central fiber  $K \times \{0\}$  lies entirely in  $K \times D_{0.9}$ .*

*In particular, because it's the only place in  $K \times D_{0.9}$  which supports any Floer generators, the output of  $u$  must also lie in the central fiber.*

*Remark 4.26.* Any holomorphic cap with Floer datum  $K^{\mathbb{C}, \mathbf{m}}$  is assumed to have incidence condition  $u(0) \in Y_\sigma$ . For such curves the condition that all inputs lie in the central fiber is vacuously true.

*Proof.* Suppose not, so that for every  $E_0$  there are infinitely many pairs  $(\mathbf{m}, u)$  which satisfy the assumptions but do not lie entirely in  $K \times D_{0.9}$ . Then we may form a sequence  $(\mathbf{m}_j, E_j, u_j)$  with  $\mathbf{m}_j \rightarrow \infty$  and  $E_j \rightarrow 0$ ,  $u_j$  satisfying the assumptions with respect to  $\mathbf{m}_j$  and  $E_j$ , and such that no  $u_j$  lies entirely in  $K \times D_{0.9}$ .

Because of the conditions on

$$\int_{\hat{M}} \max(d^\Sigma H^{\bullet, \mathbf{m}} \wedge \beta^{\bullet, \mathbf{m}}),$$

a bound on geometric energy gives a bound on topological energy. Because  $\mathcal{D}$  is compact, this combines with our bound on the output action to bound the actions of the inputs. Thus, the maps  $u_j$  satisfy a uniform maximum principle.

Now even though the Lagrangian boundary conditions become singular in the limit, the primitives of  $\lambda_M$  on all Lagrangians tend uniformly to zero, which excludes all bubbling. Thus, again using that  $\mathcal{D}$  is compact, elliptic compactness applies, so that any subsequence  $z_j$  in the domain of  $u_j$  has a subsequence  $z_{j'}$  such that  $u_{j'}$  converges locally in  $C^\infty$  on an increasing family of neighborhoods  $U_{j'}$  of  $z_{j'}$ . Because both the geometric energy and all the actions go to zero,  $u_{j'}|_{U_{j'}}$  converges to a constant map at a critical point of  $H^M$ . On the other hand, the critical locus of  $H^M$  coincides with its zero locus, so in fact  $u_{j'}|_{U_{j'}}$  converges to a point where  $H^M = 0$ .

This implies that  $\|du_j\|$  converges uniformly over the entire domain to 0, since otherwise we could find a sequence  $z_j$  near which  $u_j$  cannot converge to a constant map.

Now, for any limiting domain  $\bigcup U_{j'}$  with nonempty boundary the limit curve must lie on a point  $\Sigma L \cap (H^M)^{-1}(0)$  for some  $L$ , and all such points belong to the interior of  $K \times D_{0.9}$ . Indeed, without rescaling it is immediate from the properties of  $H^N$ , and because all rescaling factors are at least one the rescaled Hamiltonian vanishes on a strictly smaller portion of  $\Sigma L$ . Similarly, if the limiting domain has an incidence condition on  $Y_\sigma$ , then the limit curve must lie on a point  $Y_\sigma \cap (H^M)^{-1}(0)$ , and again all such points belong to the interior of  $K \times D_{0.9}$ .

The only possible limiting domain which does not fall into one of the above classes is the infinite cylinder  $\mathbb{R} \times S^1$ . Thus, for large  $j'$ , any  $u_{j'}$  which exits  $K \times D_{0.9}$  must do so along a long cylinder  $S$  with positive end near  $Y_\sigma \cap (H^M)^{-1}(0)$ . Because  $\|du_j\|$  tends uniformly to zero, we can increase  $j'$  to ensure that  $u_{j'}(S)$  is supported arbitrarily close to  $(H^M)^{-1}(0)$  and that the image of  $u_{j'}(s, \cdot)$  has small diameter for all  $s$ . Because  $H^M$  is bounded below on  $K \times D_{0.9}$  when  $|y| \geq 0.8$ , the above implies that  $u_{j'}(S)$  exits  $K \times D_{0.9}$  along  $|x| \geq 0.4$ . This contradicts (4.26) in Lemma 4.24.  $\square$

*Remark 4.27.* Let  $Y'$  be some hypersurface which lies to the right of  $Y_\sigma$  in the sense of Figure 7. Then holomorphic caps with incidence condition  $u(0) \in Y'$  must also have output in the central fiber. This is because the above argument still prevents the output from escaping to the left, while the only generators of symplectic cohomology which are supported entirely to the right of  $Y_\sigma$  live in the divisor  $D_\sigma$ . These are prohibited by the same positivity argument as in the proof of Lemma 4.12.

*Proof of Proposition 4.21.* By construction, the restriction of all Floer data to the central fiber are regular Floer data on the central fiber. In particular,  $\mathbf{K}^{\mu, \mathbf{m}}$  gives a choice of  $\mathcal{W}^{\frac{1}{\mathbf{m}}}(F)$  with respect to the normalizing Hamiltonian  $H^M|_{\hat{F} \times \{0\}}$ . Because the Hamiltonians in  $\mathbf{K}^{\mu, \mathbf{m}}$  rotate the imaginary axis counterclockwise, any generator

$$\gamma \in \text{hom}_{\mathcal{W}^{\frac{1}{\mathbf{m}}}(F)}^*(L_0, L_1)$$

which lives in  $K$  gives rise to a generator

$$\Sigma\gamma := \gamma \times \{0\} \in \text{hom}_{\mathcal{W}_\sigma^{\frac{1}{\mathbf{m}}}(M)}^*(\Sigma L_0, \Sigma L_1)$$

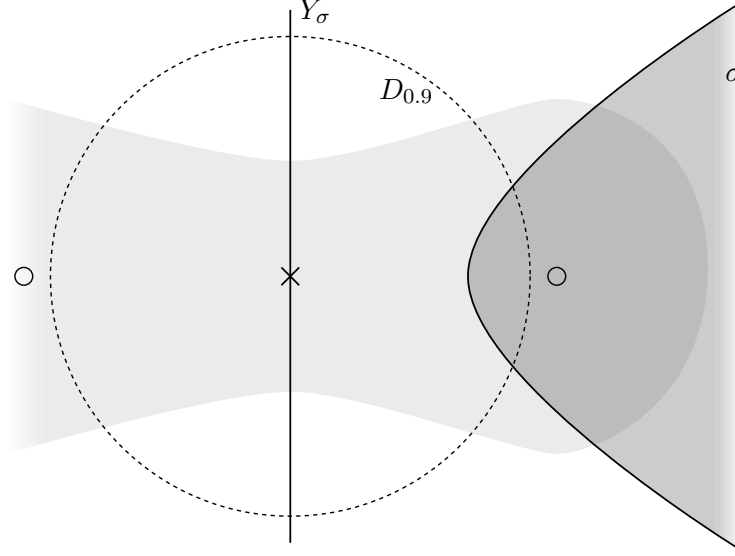


FIGURE 7. The area near the stop  $\sigma$  in  $M = M'[\sigma]$ . The large central shaded region is where  $H^M = 0$ .

of the same degree living in the central fiber. Here,  $\mathcal{W}_\sigma^{\mathbf{m}}(M)$  is the presentation of  $\mathcal{W}_\sigma(M)$  coming from  $\mathbf{K}^{\mu, \mathbf{m}}$ . Similarly, any generator  $x$  of  $SC^*(F)$  for Floer datum  $(H_t^{\mathbf{m}}, J_t^{\mathbf{m}})_{\hat{F} \times \{0\}}$  which lives in  $K$  gives rise to a generator

$$\Sigma x := x \times \{0\} \in SC_\sigma^{*+1}(M)$$

for Floer datum  $(H_t^{\mathbf{m}}, J_t^{\mathbf{m}})$  living in the central fiber. Here, the degree shift comes from the fact that in the disk-direction, the central fiber lives at a saddle point of the Hamiltonian vector field.

Now, because  $F$  is strongly nondegenerate, we may choose for all  $\mathbf{m}$  a Hochschild fundamental cycle  $e_{\mathbf{m}}$  of length at most the Hochschild length  $k$  of  $F$  such that  $\lim_{\mathbf{m} \rightarrow \infty} A_{\frac{1}{\mathbf{m}}}(e_{\mathbf{m}}) = 0$ . For large  $\mathbf{m}$ , all entries in  $e_{\mathbf{m}}$  live in  $K$ , and we write  $\Sigma e_{\mathbf{m}} \in CC_*(\mathcal{B}_\sigma^{\mathbf{m}}(\sigma))$  for the corresponding chain in the central fiber, where  $\mathcal{B}_\sigma^{\mathbf{m}}(\sigma)$  is as usual the presentation of  $\mathcal{B}_\sigma(\sigma)$  coming from  $\mathbf{K}^{\mu, \mathbf{m}}$ . For sufficiently large  $\mathbf{m}$ , Lemma 4.25 then says that all holomorphic curves contributing to  $\delta(\Sigma e_{\mathbf{m}})$  and  $\mathcal{OC}(\Sigma e_{\mathbf{m}})$  are supported entirely in  $K \times D_{0.9}$ , and that their outputs live in the central fiber. Because all of the associated Floer data are  $\Phi$ -invariant on this region and we are working in characteristic 2, the only holomorphic curves which contribute are those which live in the central fiber. This implies that

$$\delta(\Sigma e_{\mathbf{m}}) = \Sigma(\delta(e_{\mathbf{m}})) = 0 \quad \text{and} \quad \mathcal{OC}(\Sigma e_{\mathbf{m}}) = \Sigma(\mathcal{OC}(e_{\mathbf{m}})).$$

It remains to see that  $\Sigma(\mathcal{OC}(e_{\mathbf{m}}))$  is a saddle unit. For this, note that  $\mathcal{OC}(e_{\mathbf{m}})$  represents  $\mathbf{1} \in SH^0(F)$ , and that it has small action when  $\mathbf{m}$  is large. Thus, it is the sum of the Morse minima, as that is the unique such cochain. Playing the same game as for  $\mu$  and  $\mathcal{OC}$  with holomorphic caps, we see that the saddle unit is also supported in the central fiber, and there it comes from counting holomorphic caps in  $F$  with no incidence condition. The output of such caps must have small action, which implies that the saddle unit is also the sum of the Morse minima in  $F$ , as desired.  $\square$

We have in fact proved

**Corollary 4.28.** *For all sufficiently large  $\mathbf{m}$ , the Floer data  $\mathbf{K}^{\mu, \mathbf{m}}$ ,  $\mathbf{K}^{\mathcal{OC}, \mathbf{m}}$ , and  $K^{\mathbb{C}, \mathbf{m}}$  witness a Hochschild cycle  $y_{\mathbf{m}} \in CC_{1-n}(\mathcal{B}_\sigma(\sigma))$  such that  $\mathcal{OC}(y_{\mathbf{m}})$  is a saddle unit.*

*Moreover, the action of  $y_{\mathbf{m}}$  tends to zero as  $\mathbf{m}$  tends to infinity, the saddle unit  $\mathcal{OC}(y_{\mathbf{m}})$  lives in the central fiber, and  $y_{\mathbf{m}}$  is the stabilization of a Hochschild fundamental chain for the fiber of  $\sigma$ .*

This last part means that the objects making up  $y_{\mathbf{m}}$  are all of the form  $\Sigma L$  and that the morphisms making up  $y_{\mathbf{m}}$  all live in the central fiber.

**4.7. Transversality.** We now verify that the holomorphic curves of the previous section which occur in the compact part of the central fiber are automatically regular. For this, we will deduce transversality on  $M$  from transversality on  $F$ . This means that we will assume the Floer data restricts to a regular choice on the central fiber and not allow ourselves further modifications of it.

Note now that the almost complex structures are strictly split, and the Hamiltonians are split to second order. Hence, the linearized Cauchy-Riemann operator splits. In the disk component, the map is constant, which implies that the disk component of the linearized operator has no parameter associated to changing the conformal structure. This means that the disk component of the linearized operator at a map  $u: \Sigma \rightarrow \hat{M}$  landing in  $K \times \{0\}$  takes the form

$$(4.27) \quad D^{disk} \bar{\partial}(u)(\xi) = \frac{1}{2} \left( (d\xi - dX_H(u(z))(\xi) \otimes \beta) + J(z) \circ (d\xi - dX_H(u(z))(\xi) \otimes \beta) \circ j \right).$$

Now the disk component of  $X_H$  along the central fiber depends only on the local rescaling factor on  $\Sigma$ , and in particular it does not depend on  $u$ . Moreover, the Liouville vector field agrees to second order along the central fiber with the standard radially expanding one, which means that its flow is holomorphic to second order. This means that (4.27) simplifies to

$$(4.28) \quad D^{disk} \bar{\partial}(u)(\xi) = \frac{1}{2} \left( (d\xi - X_{H^{disk}}(z)(\xi) \otimes \beta) + J_{std} \circ (d\xi - X_{H^{disk}}(z)(\xi) \otimes \beta) \circ j \right),$$

where  $H^{disk}$  is a Liouville rescaling of  $a \cdot (y^2 - x^2)$ . This is manifestly independent of  $u$ , so it allows us to treat transversality by simply studying perturbed holomorphic maps to  $\mathbb{C}$  for a saddle Hamiltonian. In particular, we immediately obtain transversality for Floer strips and cylinders, and as noted in Remark 4.23 Seidel's topological argument provides transversality for  $A_\infty$  disks.

**Lemma 4.29.** *For all sufficiently large  $\mathbf{m}$ , the following holds. If  $u: \Sigma \rightarrow K \times \{0\}$  is a holomorphic curve for Floer data  $\mathbf{K}^{OC, \mathbf{m}}|_{K \times \{0\}}$  with at most  $k$  inputs, then  $u$  is regular as a map to  $\hat{M}$  if and only if it is regular as a map to  $\hat{F}$ . Here, as before,  $k$  is the Hochschild length of  $F$ .*

*Proof.* It suffices to show that with the conformal structure of  $\Sigma$  fixed,

$$D^{disk} \bar{\partial}: W^{1,p}(\Sigma, \mathbb{C}) \rightarrow L^p(\Sigma, \Omega^1 \Sigma \otimes \mathbb{C})$$

is surjective. For this, note that every input has degree 0, while the output has degree 1, so that

$$\text{ind}(D^{disk} \bar{\partial}) = \deg(x_-) - \sum_{i=1}^d \deg(\gamma_i) - \dim_{\mathbb{C}}(\mathbb{C}) = 0.$$

Hence, proving that  $D^{disk} \bar{\partial}$  is surjective is equivalent to proving that it is injective.

To that end, suppose not, so that for all  $\mathbf{m}$  there is some element  $\xi_{\mathbf{m}}$  in the kernel of  $D \bar{\partial}$  for  $H^{disk}$  and  $\beta$  coming from a Floer datum  $\mathbf{K}^{\bullet, \mathbf{m}}$ . Without loss of generality, assume that  $\|\xi_{\mathbf{m}}\|_{C^0} = 1$  for all  $\mathbf{m}$ . Now because  $H^{OC, \mathbf{m}}$  converges in  $C^\infty$  to  $H^M$  and  $X_{H^M}$  vanishes to at least third order in the disk directions along the central fiber, the linearized Hamiltonians  $H^{disk}$  tend uniformly to zero on the image of  $\xi_{\mathbf{m}}$ . This, together with the trivial identity  $E^{top}(\xi_{\mathbf{m}}) = 0$ , implies that

$$\lim_{\mathbf{m} \rightarrow \infty} E^{geom}(\xi_{\mathbf{m}}) = 0.$$

To conclude the proof, note that Lemma 4.25 applies to the maps  $\xi_{\mathbf{m}}$ . Indeed, the proof carries through to this situation verbatim, except now we need a new way to prevent the maps from escaping in the  $y$ -direction. For this, we observe that what goes up must come down: if some portion of  $\xi_{\mathbf{m}}$  reaches  $|y| \geq 0.8$  for all  $\mathbf{m}$ , then because the output is at the origin some strip or cylinder must make the return journey to  $|y| < 0.1$ . This contradicts (4.25) in Lemma 4.24. On the other hand, the conclusion  $\text{image}(\xi_{\mathbf{m}}) \in D_{0.9}$  contradicts the assumption that  $\|\xi_{\mathbf{m}}\|_{C^0} = 1$ .  $\square$



**Lemma 4.30.** *If  $u: \mathbb{C} \rightarrow K \times \{0\}$  is a holomorphic curve for Floer data  $K^{\mathbb{C}, \mathbf{m}}|_{K \times \{0\}}$ , then  $u$  is regular as a map to  $\hat{M}$  if and only if it is regular as a map to  $\hat{F}$ . In this case, the evaluation map at zero is transverse to  $Y_\sigma$ .*

*Proof.* This time we wish to show both that  $D^{disk} \bar{\partial}$  is surjective and that  $ev_0(\ker D^{disk} \bar{\partial})$  is transverse to the  $y$ -axis.

Because the output has degree 1, showing that  $D^{disk} \bar{\partial}$  is surjective amounts to showing that  $\ker D^{disk} \bar{\partial}$  is 1-dimensional. To that end, note first that for any nonconstant solution  $\xi$ , the asymptotic winding number of  $\xi$  around the origin is nonpositive, which implies that the total number of zeros of  $\xi$  is nonpositive. By positivity of intersections, this implies that  $\xi$  has no zeros. This means that  $\xi = 0$  whenever  $ev_0(\xi) = 0$ . By linearity, this implies that

$$ev_0: \ker D^{disk} \bar{\partial} \rightarrow \mathbb{C}$$

is injective.

Now there is a 1-dimensional space  $X$  of rotationally invariant solutions coming from partial increasing Morse trajectories from some point  $p \in \mathbb{R}$  to the origin. If  $X$  were not all of  $\ker D^{disk} \bar{\partial}$ , then because  $ev_0$  is injective  $\ker D^{disk} \bar{\partial}$  would be 2-dimensional. Since  $\ker D^{disk} \bar{\partial}$  comes with a continuous  $S^1$  action which fixes  $X$ , the entire  $S^1$  action must be trivial, which means all solutions are rotationally invariant. On the other hand, any rotationally invariant solution must be a partial Morse trajectory, and the asymptotic condition  $\xi(\infty) = 0$  implies that all such solutions belong to  $X$ . This shows that  $\ker D^{disk} \bar{\partial} = X$  is 1-dimensional.

For the second part, note simply that  $ev_0(X) = \mathbb{R}$  is indeed transverse to  $i\mathbb{R}$ .  $\square$

## 5. STOP REMOVAL

**5.1. A filtration on the quotient category.** To prove Theorem 4.10, we will work in Lyubashenko–Ovsienko’s model for the quotient of an  $A_\infty$ -category by a full subcategory [28], which is the  $A_\infty$  version of Drinfeld’s construction for dg-categories [12]. For an  $A_\infty$ -category  $\mathcal{A}$  and a full subcategory  $\mathcal{B} \subset \mathcal{A}$ , the quotient  $\mathcal{A}/\mathcal{B}$  is the  $A_\infty$ -category with the same objects as  $\mathcal{A}$  and whose morphism spaces are given by

$$\mathrm{hom}_{\mathcal{A}/\mathcal{B}}(L_0, L_1) = \bigoplus_{k=0}^{\infty} \bigoplus_{B_i \in \mathcal{B}} \mathrm{hom}_{\mathcal{A}}(B_k, L_1) \otimes \mathrm{hom}_{\mathcal{A}}(B_{k-1}, B_k) \otimes \cdots \otimes \mathrm{hom}_{\mathcal{A}}(L_0, B_1),$$

where for  $k = 0$  the right-hand side is just  $\mathrm{hom}_{\mathcal{A}}(L_0, L_1)$ . The grading is given by

$$\deg(\gamma^k \otimes \cdots \otimes \gamma^0) = \sum_{i=0}^k \deg(\gamma^i) - k.$$

The differential  $\mu_{\mathcal{A}/\mathcal{B}}^1$  is the bar differential, i.e.

$$\mu_{\mathcal{A}/\mathcal{B}}^1(\gamma^k \otimes \cdots \otimes \gamma^0) = \sum_{0 \leq i \leq j \leq k} \gamma^k \otimes \cdots \otimes \gamma^{j+1} \otimes \mu^{1+j-i}(\gamma^j, \dots, \gamma^i) \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0.$$

The higher operations are similar. Specifically, we have

$$\begin{aligned} \mu_{\mathcal{A}/\mathcal{B}}^d \left( (\gamma_d^{k_d} \otimes \cdots \otimes \gamma_d^0), \dots, (\gamma_1^{k_1} \otimes \cdots \otimes \gamma_1^0) \right) \\ = \sum_{\substack{0 \leq i \leq k_1 \\ 0 \leq j \leq k_d}} \gamma_d^{k_d} \otimes \cdots \otimes \mu^{i+j+d+\sum_{s=2}^{d-1} k_s}(\gamma_d^j, \dots, \gamma_1^{k_1-i}) \otimes \cdots \otimes \gamma_1^0. \end{aligned}$$

In this model,  $\mathcal{W}_\sigma(M)/\mathcal{B}_\sigma(\sigma)$  is naturally a subcategory of  $\mathcal{W}_{\sigma'}(M)/\mathcal{B}_{\sigma'}(\sigma)$ , and  $\mathcal{SR}$  is just the inclusion.

Further, we can work in  $M[\sigma]$ , so in fact we will study the inclusion

$$(5.1) \quad \mathcal{SR}_{inc}: \mathcal{W}_\sigma(M[\sigma])/\mathcal{B}_\sigma(\sigma) \hookrightarrow \mathcal{W}_{\sigma'}(M[\sigma])/\mathcal{B}_{\sigma'}(\sigma).$$

For the sake of readability, we will write  $\text{hom}_\sigma$  in place of  $\text{hom}_{\mathcal{W}_\sigma(M[\sigma])/\mathcal{B}_\sigma(\sigma)}$  and  $\text{hom}_{\sigma'}$  in place of  $\text{hom}_{\mathcal{W}_{\sigma'}(M[\sigma])/\mathcal{B}_{\sigma'}(\sigma)}$ . Similarly, we'll write  $\mu_{\sigma'}^k$  and  $\mu_\sigma^k$  for the  $A_\infty$  operations on the quotient categories.

Theorem 4.10 is equivalent to the assertion that  $\mathcal{SR}_{inc}$  is fully faithful whenever  $\sigma$  is strongly nondegenerate, which follows from the following statement:

**Proposition 5.1.** *Assume  $\sigma$  is strongly nondegenerate, and let  $L_0$  and  $L_1$  be interior Lagrangians in  $M[\sigma]$ . Then there is a retraction*

$$R: \text{hom}_{\sigma'}^*(L_0, L_1) \rightarrow \text{hom}_\sigma^*(L_0, L_1)$$

with the following property:

$$(5.2) \quad \begin{array}{l} \text{For any finite dimensional subcomplex } C \subset \text{hom}_{\sigma'}^*(L_0, L_1), \text{ there is a chain map} \\ \text{from } \text{hom}_{\sigma'}^*(L_0, L_1) \text{ to itself which is homotopic to the identity and agrees with } R \\ \text{on } C. \end{array}$$

As a topological analogy, this says that  $R$  is a deformation retraction on compact subsets. In particular, it is a quasi-isomorphism.

To construct  $R$  and prove that it satisfies (5.2), we will build an increasing filtration on the morphism spaces of  $\mathcal{W}_{\sigma'}(M[\sigma])/\mathcal{B}_{\sigma'}(\sigma)$  and a homotopy that moves us down in the filtration.

**Definition 5.2.** Consider the lexicographic order on  $\mathbb{N}^2$ , namely  $(\mathbf{n}, \mathbf{m}) < (\mathbf{n}', \mathbf{m}')$  if  $\mathbf{n} < \mathbf{n}'$  or both  $\mathbf{n} = \mathbf{n}'$  and  $\mathbf{m} < \mathbf{m}'$ . This has order type  $\omega^2$ , so in particular it is a well-ordering.

Define  $A_{\mathbf{n}, \mathbf{m}}^* \subset \text{hom}_{\sigma'}^*(L_0, L_1)$  to be the graded vector subspace generated by

$$\left\{ \gamma^k \otimes \cdots \otimes \gamma^0 \mid \left( \sum_{i=0}^k n_\sigma(\gamma^i), k \right) < (\mathbf{n}, \mathbf{m}) \right\}.$$

Then  $A_{\mathbf{n}, \mathbf{m}}^*$  is a subcomplex, and  $A_{\mathbf{n}, \mathbf{m}}^* \subset A_{\mathbf{n}', \mathbf{m}'}^*$  whenever  $(\mathbf{n}, \mathbf{m}) < (\mathbf{n}', \mathbf{m}')$ . This means that the  $A_{\mathbf{n}, \mathbf{m}}^*$  form an exhausting filtration of  $\text{hom}_{\sigma'}^*(L_0, L_1)$ , which we call the **main filtration**. Note that  $A_{1,0}^* = \text{hom}_\sigma^*(L_0, L_1)$ .

To prove Proposition 5.1, we will construct a map of graded vector spaces

$$\Delta_y: \text{hom}_{\sigma'}^*(L_0, L_1) \rightarrow \text{hom}_{\sigma'}^{*-1}(L_0, L_1)$$

such that

$$(5.3) \quad R_y := \text{id} + \mu_{\sigma'}^1 \Delta_y + \Delta_y \mu_{\sigma'}^1$$

is the identity on  $A_{1,0}^* = \text{hom}_\sigma^*(L_0, L_1)$  and strictly decreases the filtration on  $A_{\mathbf{n}, \mathbf{m}}^*$  for  $(\mathbf{n}, \mathbf{m}) > (1, 0)$ . Since the filtration is well-ordered, the sequence

$$\gamma, R_y(\gamma), R_y(R_y(\gamma)), \dots$$

stabilizes, and hence the infinite iterate

$$R_y^\infty: \text{hom}_{\sigma'}^*(L_0, L_1) \rightarrow \text{hom}_{\sigma'}^*(L_0, L_1)$$

is well defined.

**Lemma 5.3.** *For an element  $\gamma \in \text{hom}_{\sigma'}^*(L_0, L_1)$ , let  $(\mathbf{n}, \mathbf{m})(\gamma)$  be the smallest pair for which  $\gamma \in A_{\mathbf{n}, \mathbf{m}}^*$ . Then for any  $R_y$  such that*

- (1)  $R_y(\gamma) = \gamma$  for  $\gamma \in A_{1,0}^*$ , and
- (2)  $(\mathbf{n}, \mathbf{m})(R_y(\gamma)) < (\mathbf{n}, \mathbf{m})(\gamma)$  whenever  $(\mathbf{n}, \mathbf{m})(\gamma) > (1, 0)$ ,

$R_y^\infty$  satisfies the conditions on  $R$  in Proposition 5.1.

*Proof.* Since the iterates of  $R_y$  stabilize for any element, they stabilize after finitely many steps on any finite dimensional subspace. Thus, for any finite dimensional subcomplex  $C$ , there is some  $n$  such that  $R_y^n$  agrees with  $R_y^\infty$  on  $C$ . Since  $R_y$  is homotopic to the identity, so is the finite iterate  $R_y^n$ .  $\square$

**5.2. Coproduct disks.** The map  $\Delta_y$  giving rise to  $R_y$  will be given by a certain coproduct operation, which will come as always from counts of holomorphic disks. We describe these now.

**Definition 5.4.** For nonnegative integers  $d, k, l$ , let  $\mathcal{R}^{d;k,l}$  be the abstract **moduli space of coproduct disks**. These are disks  $\Sigma$  with  $d+k+l+3$  boundary punctures labeled in counterclockwise order as follows.

- (1)  $\zeta_i$ , for  $i$  increasing from  $-k$  to  $+l$ . These will eventually be equipped with positive strip-like ends.
- (2)  $\zeta_a$ , which will eventually be equipped with a negative strip-like end.
- (3)  $\zeta^j$  for  $j$  increasing from 1 to  $d$ . These will eventually be equipped with positive strip-like ends.
- (4)  $\zeta_b$ , which will eventually be equipped with a negative strip-like end.

The punctures  $\zeta_0$ ,  $\zeta_a$ , and  $\zeta_b$  are considered **distinguished points**. Observe that any disk with  $n+3$  punctures can be made into an element of some  $\mathcal{R}^{d;k,l}$  with  $d+k+l = n$  by specifying and labeling three distinguished points. The compactified moduli space  $\overline{\mathcal{R}}^{d;k,l}$  is diffeomorphic to the associahedron  $\overline{\mathcal{R}}^{(d+k+l+2)+1}$ , where the identification can be taken to match  $\zeta_0$  with  $\zeta_0$ . The codimension  $r$  boundary faces of  $\overline{\mathcal{R}}^{d;k,l}$  are identified with products of some lower-dimensional  $\mathcal{R}^{d';k',l'}$  with  $r$  lower-dimensional associahedra inductively as follows.

For a codimension 1 face, a point corresponds to a pair of disks identified at new boundary punctures  $\tilde{\zeta}$ , and we may look at the induced labels of boundary punctures on these two disks. One of these disks, which we call  $\Sigma_0$ , contains two or three distinguished points, while the other disk  $\Sigma_1$ , contains one or zero. In each case  $\Sigma_0$  will be taken to lie in  $\mathcal{R}^{d';k',l'}$ , but there are several ways that this can happen.

- (1) The first possibility is that  $\Sigma_0$  contains all three distinguished points. In this case it is identified with an element of  $\mathcal{R}^{d';k',l'}$  by matching up the distinguished points.  $\Sigma_1$  is identified with a point of  $\mathcal{R}^{m+1}$  by taking  $\tilde{\zeta}$  to be the root.
- (2) The second possibility, similar to the first, is that  $\Sigma_0$  contains  $\zeta_a$  and  $\zeta_b$ , while  $\Sigma_1$  contains  $\zeta_0$ . In this case  $\Sigma_0$  is identified with the element of  $\mathcal{R}^{d';k',l'}$  which has  $\zeta_a$  and  $\zeta_b$  in the same place and  $\zeta_0$  in the position of  $\tilde{\zeta}$ .  $\Sigma_1$  is again identified with a point of  $\mathcal{R}^{m+1}$  by taking  $\tilde{\zeta}$  to be the root. By remembering the distinguished point  $\zeta_0 \in \Sigma_1$ , we may upgrade it to an element of  $\mathcal{R}^{m+1,i}$  for some  $i$ .
- (3) The third and fourth possibilities are that  $\Sigma_1$  contains  $\zeta_a$  or  $\zeta_b$ . We assume that  $\zeta_a \in \Sigma_1$ , as the other situation is strictly similar. In this case, for  $\Sigma_0$ ,  $\tilde{\zeta}$  takes the place of  $\zeta_a$  as the third distinguished point, while  $\Sigma_1$  is identified with an element of  $\mathcal{R}^{m+1}$  by setting  $\zeta_a$  to be the root.

For a higher codimension face, we obtain a decomposition by following a sequence of faces, each of which has codimension 1 in the previous. To see that the decomposition is unique, note that an element  $\Sigma$  of the boundary of  $\overline{\mathcal{R}}^{d;k,l}$  is a disk with boundary nodes described by a tree  $T$ . If at least two distinguished points of  $\Sigma$  live on the same component, then that component is the one identified with an element of  $\mathcal{R}^{d';k',l'}$ . Otherwise, there is a unique vertex  $v$  of  $T$  such that every path from  $v$  to a vertex containing a distinguished point leaves  $v$  along a different edge. The component of  $\Sigma$  identified with a point of  $\mathcal{R}^{d';k',l'}$  is the one corresponding to  $v$ .

As indicated, a collection of strip-like ends for a disk  $\Sigma \in \mathcal{R}^{d;k,l}$  consists of a positive strip-like end at each puncture  $\zeta_i$  and  $\zeta^j$ , along with a negative strip-like end at each of  $\zeta_a$  and  $\zeta_b$ , such that the images of the ends are pairwise disjoint. A **universal choice** of strip-like ends for  $\mathcal{R}^{d;k,l}$  consists of, for all  $d, k, l \geq 0$ , a collection of strip-like ends for each  $\Sigma \in \mathcal{R}^{d;k,l}$  which varies smoothly over  $\mathcal{R}^{d;k,l}$  and agrees near the boundary with the collection of ends induced by gluing. As with associahedra, a universal choice of strip-like ends for  $\mathcal{R}^{d;k,l}$  can be constructed inductively, and we fix one once and for all.

Similarly, a **universal and conformally consistent** choice of Floer data for  $\mathcal{R}^{d;k,l}$  consists of, for all  $d, k, l \geq 0$ , a Floer datum for each  $\Sigma \in \mathcal{R}^{d;k,l}$  which varies smoothly over  $\mathcal{R}^{d;k,l}$ , and which additionally satisfies the asymptotic consistency condition of Definition 3.10 with  $\mathcal{R}^{d+1}$  replaced by  $\mathcal{R}^{d;k,l}$ . Let  $\mathcal{K}_\Delta(M[\sigma])$  denote the space of all universal and conformally consistent choices Floer data for  $\mathcal{R}^{d;k,l}$ .

For any  $\mathbf{K}_\Delta \in \mathcal{K}_\Delta(M[\sigma])$ , we obtain a perturbed Cauchy-Riemann operator. Given Lagrangians  $L_0, \dots, L_d$  and  $B_{-k-1}, \dots, B_l$  and chords

$$(5.4) \quad \begin{aligned} \gamma_i &\in \mathcal{X}(B_{i-1}, B_i) & \gamma_a &\in \mathcal{X}(L_0, B_l) \\ \gamma^j &\in \mathcal{X}(L_{j-1}, L_j) & \gamma_b &\in \mathcal{X}(B_{-k-1}, L_d), \end{aligned}$$

we can consider the space  $\mathcal{R}^{d;k,l}(\vec{\gamma}^*, \vec{\gamma}_*; \gamma_b, \gamma_a)$ , where  $\vec{\gamma}^*$  and  $\vec{\gamma}_*$  are the tuples  $(\gamma^d, \dots, \gamma^1)$  and  $(\gamma_l, \dots, \gamma_{-k})$ , respectively. This consists of all maps  $u: \Sigma \rightarrow \widehat{M[\sigma]}$ , with  $\Sigma$  ranging over  $\mathcal{R}^{d;k,l}$ , satisfying (3.15) with

$$\begin{aligned} u(\zeta_i) &= (\phi^{\tau_i})^* \gamma_i & u(\zeta_a) &= (\phi^{\tau_a})^* \gamma_a \\ u(\zeta^j) &= (\phi^{\tau^j})^* \gamma^j & u(\zeta_b) &= (\phi^{\tau_b})^* \gamma_b \end{aligned}$$

and with the appropriate boundary conditions, where  $\tau_i$  is the rescaling factor assigned to  $\zeta_i$ , and similarly with  $\tau^j$ ,  $\tau_a$ , and  $\tau_b$ . As usual, Lemma A.8 tells us that the images of such  $u$  are all contained in a fixed compact subset of  $\widehat{M}$ , so Gromov compactness gives  $\mathcal{R}^{d;k,l}(\vec{\gamma}^*, \vec{\gamma}_*; \gamma_b, \gamma_a)$  a natural compactification  $\overline{\mathcal{R}}^{d;k,l}(\vec{\gamma}^*, \vec{\gamma}_*; \gamma_b, \gamma_a)$  whose new points are broken configurations consisting of one element of  $\mathcal{R}^{d';k',l'}((\vec{\gamma}^*)', (\vec{\gamma}_*)'; \gamma'_b, \gamma'_a)$  for some  $d' \leq d$ ,  $k' \leq k$ , and  $l' \leq l$ , along with disks contributing to the  $A_\infty$  structure. We list those configurations with exactly two nonconstant components.

$$(5.5a) \quad \begin{aligned} &\mathcal{R}^{d;k+1-m,l}(\vec{\gamma}^*, (\gamma_l, \dots, \gamma_{i+m}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_{-k}); \gamma_b, \gamma_a) & 2 \leq m \leq -i \leq k \\ &\times \mathcal{R}^{m+1}(\gamma_{i+m-1}, \dots, \gamma_i; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}(B_{i-1}, B_{i+m-1}) \end{aligned}$$

$$(5.5b) \quad \begin{aligned} &\mathcal{R}^{d;k,l+1-m}(\vec{\gamma}^*, (\gamma_l, \dots, \gamma_{i+m}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_{-k}); \gamma_b, \gamma_a) & 2 \leq m \leq 1+l-i \leq l \\ &\times \mathcal{R}^{m+1}(\gamma_{i+m-1}, \dots, \gamma_i; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}(B_{i-1}, B_{i+m-1}) \end{aligned}$$

$$(5.5c) \quad \begin{aligned} &\mathcal{R}^{d;k+i,l-i+1-m}(\vec{\gamma}^*, (\gamma_l, \dots, \gamma_{i+m}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_{-k}); \gamma_b, \gamma_a) & -k \leq i \leq 0 \\ &\times \mathcal{R}^{m+1}(\gamma_{i+m-1}, \dots, \gamma_i; \tilde{\gamma}) & \min\{2, 1-i\} \leq m \leq l-i+1 \\ & & \tilde{\gamma} \in \mathcal{X}(B_{i-1}, B_{i+m-1}) \end{aligned}$$

$$(5.5d) \quad \begin{aligned} &\mathcal{R}^{d;k,l}(\vec{\gamma}^*, (\gamma_l, \dots, \gamma_{i+1}, \tilde{\gamma}, \gamma_{i-1}, \dots, \gamma_{-k}); \gamma_b, \gamma_a) & -k \leq i \leq l \\ &\times \mathcal{R}(\gamma_i; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}(B_{i-1}, B_i) \end{aligned}$$

$$(5.5e) \quad \begin{aligned} &\mathcal{R}^{d+1-m;k,l}((\gamma^d, \dots, \gamma^{i+m}, \tilde{\gamma}, \gamma^{i-1}, \dots, \gamma^1), \vec{\gamma}_*; \gamma_b, \gamma_a) & 2 \leq m \leq d+1-i \leq d \\ &\times \mathcal{R}^{m+1}(\gamma^{i+m-1}, \dots, \gamma^i; \tilde{\gamma}) & \tilde{\gamma} \in \mathcal{X}(L_{i-1}, L_{i+m-1}) \end{aligned}$$

$$(5.5f) \quad \begin{aligned} & \mathcal{R}^{d;k,l}((\gamma^d, \dots, \gamma^{i+1}, \tilde{\gamma}, \gamma^{i-1}, \dots, \gamma^1), \vec{\gamma}_\star; \gamma_b, \gamma_a) \\ & \quad \times \mathcal{R}(\gamma^i; \tilde{\gamma}) \end{aligned} \quad \begin{aligned} & 1 \leq i \leq d \\ & \tilde{\gamma} \in \mathcal{X}(L_{i-1}, L_i) \end{aligned}$$

$$(5.5g) \quad \begin{aligned} & \mathcal{R}^{i+j+1+1}(\gamma_l, \dots, \gamma_{l-j+1}, \tilde{\gamma}, \gamma^i, \dots, \gamma^1; \gamma_a) \\ & \times \mathcal{R}^{d-i;k,l-j}((\gamma^d, \dots, \gamma^{i+1}), (\gamma_{l-j}, \dots, \gamma_{-k}); \gamma_b, \tilde{\gamma}) \end{aligned} \quad \begin{aligned} & 0 \leq i \leq d \\ & \max\{0, 1-i\} \leq j \leq l \\ & \tilde{\gamma} \in \mathcal{X}(L_i, B_{l-j}) \end{aligned}$$

$$(5.5h) \quad \mathcal{R}(\tilde{\gamma}; \gamma_a) \times \mathcal{R}^{d;k,l}(\vec{\gamma}_\star, \vec{\gamma}_\star; \gamma_b, \tilde{\gamma}) \quad \tilde{\gamma} \in \mathcal{X}(L_0, B_l)$$

$$(5.5i) \quad \begin{aligned} & \mathcal{R}^{i+j+1+1}(\gamma^d, \dots, \gamma^{d+1-i}, \tilde{\gamma}, \gamma_{j-k-1}, \dots, \gamma_{-k}; \gamma_b) \\ & \times \mathcal{R}^{d-i;k-j,l}((\gamma^{d-i}, \dots, \gamma^1), (\gamma_l, \dots, \gamma_{j-k}); \tilde{\gamma}, \gamma_a) \end{aligned} \quad \begin{aligned} & 0 \leq i \leq d \\ & \max\{0, 1-i\} \leq j \leq k \\ & \tilde{\gamma} \in \mathcal{X}(B_{j-k-1}, L_{d-i}) \end{aligned}$$

$$(5.5j) \quad \mathcal{R}(\tilde{\gamma}; \gamma_b) \times \mathcal{R}^{d;k,l}(\vec{\gamma}_\star, \vec{\gamma}_\star; \tilde{\gamma}, \gamma_a) \quad \tilde{\gamma} \in \mathcal{X}(B_{-k-1}, L_d)$$

This looks like a lot, but the first four are just different ways that an  $A_\infty$  disk can break off on the “subscript” side, of which the first three differ only in the placement of the marked input. The others, in pairs, describe the possible breakings of an  $A_\infty$  disk on the “superscript” side, at  $\zeta_a$ , and at  $\zeta_b$ .

**Lemma 5.5.** *There is a dense subset  $\mathcal{K}_{\Delta, \text{reg}}(M[\sigma]) \subset \mathcal{K}_\Delta(M[\sigma])$  such that, for every universal choice  $\mathbf{K}_\Delta \in \mathcal{K}_{\Delta, \text{reg}}(M[\sigma])$ , the following hold.*

- (1) *For any Lagrangians  $L_0, \dots, L_d$  and  $B_{-k-1}, \dots, B_l$  and any chords  $\gamma_j, \gamma^j$ ,  $\gamma_a$ , and  $\gamma_b$  as in 5.4,  $\mathcal{R}^{d;k,l}(\vec{\gamma}_\star, \vec{\gamma}_\star; \gamma_b, \gamma_a)$  is a smooth manifold of dimension*

$$\deg(\gamma_a) + \deg(\gamma_b) - \sum_{r=d}^d \deg(\gamma^r) - \sum_{s=-k}^l \deg(\gamma_s) + d + k + l - n.$$

- (2) *Whenever*

$$n_\sigma(\gamma_a) + n_\sigma(\gamma_b) > \sum_{r=1}^d n_\sigma(\gamma^r) + \sum_{s=-k}^l n_\sigma(\gamma_s),$$

$\mathcal{R}^{d;k,l}(\vec{\gamma}_\star, \vec{\gamma}_\star; \gamma_b, \gamma_a)$  *is empty.*

- (3) *If*

$$\deg(\gamma_a) + \deg(\gamma_b) - \sum_{r=1}^d \deg(\gamma^r) - \sum_{s=-k}^l \deg(\gamma_s) = n - d - k - l,$$

*then  $\mathcal{R}^{d;k,l}(\vec{\gamma}_\star, \vec{\gamma}_\star; \gamma_b, \gamma_a)$  is compact.*

- (4) *If*

$$\deg(\gamma_a) + \deg(\gamma_b) - \sum_{r=1}^d \deg(\gamma^r) - \sum_{s=-k}^l \deg(\gamma_s) = n - d - k - l + 1,$$

*then  $\mathcal{R}^{d;k,l}(\vec{\gamma}_\star, \vec{\gamma}_\star; \gamma_b, \gamma_a)$  admits a Gromov compactification as a compact topological 1-manifold with boundary, and its boundary is in natural bijection with the configurations (5.5).*

□

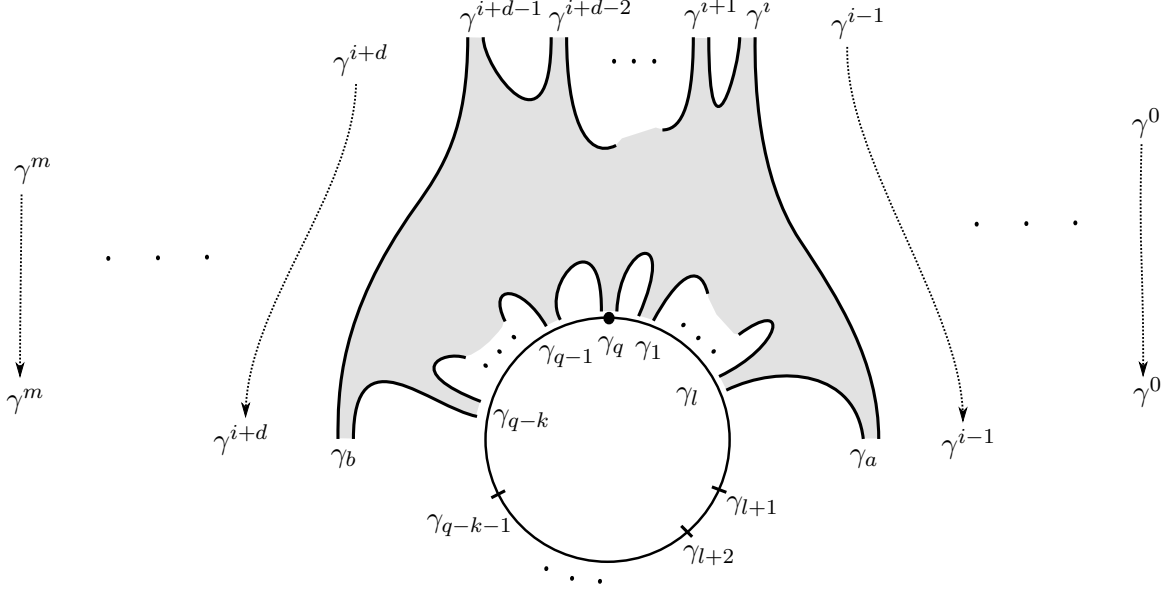


FIGURE 8.

**5.3. The main homotopy.** We are now prepared to begin constructing the operation  $\Delta_y$  which is used in the definition of the basic retraction  $R_y$  in (5.3). Concretely, we will give a formula for  $\Delta_v$  for  $v = \gamma_q \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_\sigma(\sigma))$  a low-action stabilization (cf. Corollary 4.28) and extend to general low-action stabilizations in  $CC_*(\mathcal{B}_\sigma(\sigma))$  by linearity. For now, we define a coproduct operation  $\Delta_v^0$ . This is the **main homotopy**. Later on, we will define a second operation  $h_v$  and set  $\Delta_v = \Delta_v^0 + h_v$ .

For all  $\mathbf{m} \in \mathbb{N}$ , choose as in Section 4.6 a regular Floer datum  $\mathbf{K}_\Delta^{\mathbf{m}}$  for the coproduct which satisfies the analogs of Lemmas 4.24 and 4.25, but for which the Hamiltonian is not necessarily  $\Phi$ -invariant on  $K \times D_{0.9}$ . Denote the resulting moduli spaces by  $\mathcal{R}_{\mathbf{m}}^{d;k,l}(\vec{\gamma}^*, \vec{\gamma}_*; \gamma_b, \gamma_a)$ .

For a morphism  $\gamma = \gamma^m \otimes \cdots \otimes \gamma^0 \in \text{hom}_{\sigma'}^*(L_0, L_1)$  define

$$(5.6) \quad \Delta_{v,\mathbf{m}}^0(\gamma) = \sum_{\substack{0 \leq i \leq m+1 \\ 0 \leq d \leq m+1-i \\ k+l \leq q \\ n_\sigma(\gamma^r)=0 \forall r < i}} \sum_{\substack{\gamma_a, \gamma_b \\ n_\sigma(\gamma_a)=0 \\ \deg(\gamma_a) + \deg(\gamma_b) = \sum_{r=i}^{i+d-1} \deg(\gamma^r) + \sum_{s=q-k}^l \deg(\gamma_s) + n - d - k - l}} \# \mathcal{R}_{\mathbf{m}}^{d;k,l}((\gamma^{i+d-1}, \dots, \gamma^i), (\gamma_l, \dots, \gamma_{q-k}); \gamma_b, \gamma_a) \cdot \hat{\gamma},$$

where tuples with increasing or nonexistent indices are the empty tuple  $()$ , and

$$(5.7) \quad \hat{\gamma} := \gamma^m \otimes \cdots \otimes \gamma^{i+d} \otimes \gamma_b \otimes \gamma_{q-k-1} \otimes \cdots \otimes \gamma_{l+1} \otimes \gamma_a \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0$$

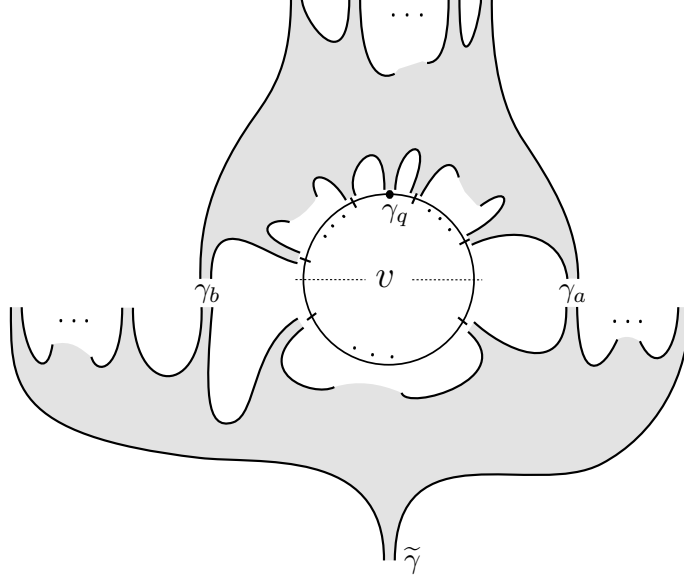
is required to be composable. See Figure 8. Again the indices for the Hochschild chain  $v$  are cyclically ordered. A straightforward calculation gives that  $\Delta_{v,\mathbf{m}}^0$  is homogeneous of degree  $\deg(v) + n - 2$ . In particular,  $\Delta_{v,\mathbf{m}}^0$  has degree -1 when  $\deg(v) = 1 - n$ , which is precisely the grading which appears in Definition 4.18.

*Remark 5.6.* Note that  $\Delta_{v,\mathbf{m}}^0$  is in general poorly defined, as (5.6) in principle allows contributions with  $d = 0$  and  $i = 0$  or  $m + 1$ . Such terms would be elements of

$$\text{hom}(B_m, L_1) \otimes \cdots \otimes \text{hom}(L_0, B_1) \otimes \text{hom}(B_{q-k}^v, L_0) \otimes \cdots \otimes \text{hom}(L_0, B_{l+1}^v)$$

or

$$\text{hom}(B_{q-k}^v, L_1) \otimes \cdots \otimes \text{hom}(L_1, B_{l+1}^v) \otimes \text{hom}(B_m, L_1) \otimes \cdots \otimes \text{hom}(L_0, B_1),$$

FIGURE 9. A formal annulus contributing to  $A_v$ .

which are not components of  $\text{hom}_{\sigma'}(L_0, L_1)$ . However, for large  $\mathbf{m}$  and  $v$  a low-action stabilization, these terms can be excluded either for energy reasons or by appealing to the analog of Lemma 4.25. In particular,  $\Delta_{v,\mathbf{m}}^0$  is well defined when  $\mathbf{m}$  is large and  $v$  is a term in the stabilization of a low-action Hochschild fundamental cycle. We will suppress the  $\mathbf{m}$  and write  $\Delta_v^0$  to denote  $\Delta_{v,\mathbf{m}}^0$  for large but indeterminate  $\mathbf{m}$ . Later on we will need to increase  $\mathbf{m}$  whenever we introduce a new moduli space. This only happens finitely many times, so it is not a problem, and we will do it implicitly without change to the notation.

*Remark 5.7.* The key condition here is that, in the output, all chords starting with  $\gamma_a$  must have crossing number zero with  $\sigma$ . This is what allows the resulting chain map to interact with the intersection filtration. In particular, we will see that  $\mu_{\sigma'}^1 \Delta_y^0 + \Delta_y^0 \mu_{\sigma'}^1$  is nontrivial precisely at the smallest  $r$  such that  $n_\sigma(\gamma^r) \neq 0$ , where it is homotopic to the identity up to terms lower in the main filtration.

We begin by examining the configurations of holomorphic disks which appear in  $\Delta_v^0 \mu_{\sigma'}^1(\gamma)$ . These come in two types.

(5.8) The first type occurs when the superscript inputs for  $\Delta_v^0$  do not include the output for  $\mu_{\sigma'}^1$ . In this case, there are two components which are disjoint and do not want to glue together.

(5.9) The second type occurs when the superscript inputs for  $\Delta_v^0$  do include the output for  $\mu_{\sigma'}^1$ . In this case, the configuration is a broken disk of the form (5.5e) or (5.5f).

Next, we examine the configurations of holomorphic disks which appear in  $\mu_{\sigma'}^1 \Delta_v^0(\gamma)$ . These come in five types.

(5.10) The first type occurs when the inputs for  $\mu_{\sigma'}^1$  do not include any of  $\gamma_a$ ,  $\gamma_b$ , or the  $\gamma_i$  coming from the unused components of  $v$ . In this case, there are two components which are disjoint and do not want to glue together. These configurations are exactly the same as those in (5.8), so their contributions to  $\Delta_v^0 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 \Delta_v^0$  cancel.

(5.11) The second type occurs when the inputs for  $\mu_{\sigma'}^1$  consist of one or more of the  $\gamma_i$  coming from the unused components of  $v$ . In this case, the configuration consists of two disjoint disks, each of which uses different portions of the Hochschild chain  $v$ .

(5.12) The third type occurs when the inputs for  $\mu_{\sigma'}^1$  include  $\gamma_a$  but not  $\gamma_b$ . In this case, the configuration is a broken disk of the form (5.5g) or (5.5h).

(5.13) The fourth type occurs when the inputs for  $\mu_{\sigma'}^1$  include  $\gamma_b$  but not  $\gamma_a$ . In this case, the configuration is a broken disk of the form (5.5i) or (5.5j).

(5.14) The fifth type occurs when the inputs for  $\mu_{\sigma'}^1$  include both  $\gamma_a$  and  $\gamma_b$ . In this case, the configuration is formally an annulus with two nodes, see Figure 9. The outside of this annulus is labeled with some substring of  $\gamma$  and an output chord  $\tilde{\gamma}$ , while the inside is labeled with the entire Hochschild chain  $v$ . Let

$$A_v: \text{hom}_{\sigma'}^*(L_0, L_1) \rightarrow \text{hom}_{\sigma'}^{*+\deg(v)+n-1}(L_0, L_1)$$

be the linear map obtained by counting only such annuli.

We claim that, modulo terms which decrease the main filtration,

$$(5.15) \quad \Delta_v^0 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 \Delta_v^0 = \Delta_{\delta v}^0 + A_v.$$

For this, it suffices to consider those contributions to  $\Delta_v^0 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 \Delta_v^0$  which come from configurations which avoid  $D_\sigma$ . Considering only such configurations, we want to see that the terms coming from (5.8)-(5.13) add up to the corresponding portion of  $\Delta_{\delta v}^0$ . To do so, note that there are two types of configurations which contribute to  $\Delta_{\delta v}^0$ . The first type occurs when the output of the  $A_\infty$  disk contributing to  $\delta v$  is not an input of  $\mathcal{R}^{d;k,l}(\tilde{\gamma}^*, \tilde{\gamma}_*; \gamma_b, \gamma_a)$ , so instead it appears in  $\hat{\gamma}$ . In this case, the configuration is precisely what is counted in (5.11). The remaining type of configuration occurs when the output of the  $A_\infty$  disk contributing to  $\delta v$  is a component of  $\tilde{\gamma}_*$ . In this case, the broken configuration is one of (5.5a)-(5.5d). Because the spaces in (5.5) form the boundary of a compact 1-manifold, we are left with terms coming from configurations of the form (5.5e)-(5.5j). On the other hand, for spaces of coproduct disks which do not intersect  $D_\sigma$ , the condition that  $n_\sigma(\gamma_a) = 0$  is preserved under breaking off an  $A_\infty$  disk. Thus, the operation coming from these configurations coincides precisely with the sum of the remaining terms (5.9), (5.12), and (5.13).

Since we will eventually be interested in replacing  $v$  with a closed chain  $y$ , we can ignore the  $\delta v$  term in (5.15). Moreover, since our goal is to show that  $R_y$  satisfies the conditions of Lemma 5.3, we may ignore all terms of  $\Delta_v^0 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 \Delta_v^0$  which strictly decrease the main filtration. Because all operations involved satisfy positivity of intersections and  $y$  is made up of chords with  $n_\sigma = 0$ , the only way in which they can fail to decrease the filtration is by failing to decrease the length of a generator  $\gamma^m \otimes \cdots \otimes \gamma^0$ . For the annulus term  $A_y$ , this only happens when the broken annuli are labeled with zero or one  $\gamma^j$  input. Write  $A_y = \sum_{\nu, \mu} A_y^{\nu, \mu}$ , where  $A_y^{\nu, \mu}$  is the operation coming from those broken holomorphic annuli with  $\mu$  superscript inputs and intersection number  $\nu$  with  $D_\sigma$ . This means, for  $A_y^{\nu, \mu}(\gamma) \neq 0$ , we have

$$(\mathbf{n}, \mathbf{m})(A_y^{\nu, \mu}(\gamma)) = (\mathbf{n}, \mathbf{m})(\gamma) + (-\nu, 1 - \mu).$$

We conclude

**Lemma 5.8.** *Let  $y$  be a closed, low-action element of  $CC_*(\mathcal{B}_\sigma(\sigma))$ . Then, up to terms which decrease the main filtration,  $\Delta_y^0 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 \Delta_y^0 = A_y^{0,0} + A_y^{0,1}$ .*  $\square$



**5.4. Closed-open maps.** The eventual objective will be to show that  $A_y^{0,0} + A_y^{0,1}$  is homotopic to a closed-open operation  $\mathcal{CO}_x^{filt}$  depending on a cochain  $x \in SC_\sigma^*(M[\sigma])$ , where it will turn out that  $x = \mathcal{OC}(y)$ . We now construct this operation.

Let  $\mathcal{R}_1^{0+1}$  be the singleton set containing a disk  $D_1^{0,1}$  with one interior puncture  $\zeta_+$  and one boundary puncture  $\zeta_0$ . Up to biholomorphism, there is a unique such disk. Equip  $\zeta_0$  with a negative strip-like end  $\epsilon_0$  and  $\zeta_+$  with a positive cylindrical end  $\epsilon_+$ . As with the punctured disks giving rise to  $\mathcal{OC}$ , we ask that  $\epsilon_+$  has a very special form. Specifically, in the holomorphic coordinates on  $D_1^{0,1}$  where  $\text{int}(\Sigma) = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$  and  $\zeta_0 = -1$ , we require that

$$(5.16) \quad \epsilon_+(s, t) = ae^{-2\pi(s+it)} \quad \text{with } a \in \mathbb{R} \text{ positive.}$$

Going up in dimension, let  $\mathcal{R}_1^{1+1}$  be the space of disks with one interior puncture  $\zeta_+$  and two boundary punctures  $\zeta_0$  and  $\zeta_1$ . The corresponding compactified moduli space  $\overline{\mathcal{R}}_1^{1+1}$  is

$$(5.17) \quad \overline{\mathcal{R}}_1^{1+1} = (\mathcal{R}^{2+1} \times \mathcal{R}_1^{0+1}) \amalg \mathcal{R}_1^{1+1} \amalg (\mathcal{R}^{2+1} \times \mathcal{R}_1^{0+1}),$$

where the two broken configurations correspond to the two ways of attaching the negative end of  $D_1^{0,1}$  to one of the two positive ends of the unique disk  $\Sigma^{2+1} \in \mathcal{R}^{2+1}$ . Choose smooth, disjoint  $\mathcal{R}_1^{1+1}$ -parametrized families of positive strip-like ends  $\epsilon_1$  for  $\zeta_1$ , negative strip-like ends  $\epsilon_0$  for  $\zeta_0$ , and positive cylindrical ends  $\epsilon_+$  for  $\zeta_+$ , which satisfy the compatibility conditions

- (1) In the gluing charts of the form  $[0, a) \times \mathcal{R}^{2+1} \times \mathcal{R}_1^{0+1}$  with gluing length  $\ell = e^{\frac{1}{\rho}}$ , where  $\rho \in [0, a)$ , the families of ends agree to infinite order at  $\rho = 0$  with those induced by gluing.
- (2) For all  $\Sigma \in \mathcal{R}_1^{1+1}$ , in the holomorphic coordinates on  $\Sigma$  where the interior of  $\Sigma$  is the punctured disk  $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$  and  $\zeta_0 = -1$ ,  $\epsilon_+$  satisfies (5.16).

Unlike with open-closed maps, for careful choices of ends elsewhere the agreement to infinite order could be strengthened to agreement in a neighborhood of the boundary, but there is no benefit to doing so. One could also extend the above and construct a map to Hochschild cohomology as in [19], but in our application the higher terms would reduce the main filtration, so we ignore them.

A **conformally consistent** choice of Floer data for the closed-open maps consists of a Floer datum on  $D_1^{0,1}$ , along with a Floer datum on  $\Sigma$  for each  $\Sigma \in \mathcal{R}_1^{1+1}$  varying smoothly over  $\mathcal{R}_1^{1+1}$ , and such that near  $\partial\overline{\mathcal{R}}_1^{1+1}$  it agrees to infinite order with the conformal class of not-quite Floer data determined by gluing. Denote by  $\mathcal{K}^{\mathcal{CO}}(M[\sigma])$  the space of conformally consistent choices of Floer data for the closed-open maps.

Given  $\mathbf{K} \subset \mathcal{K}^{\mathcal{CO}}(M[\sigma])$ , we can consider the resulting holomorphic curves. Given Lagrangian labels  $L_i$  and asymptotic ends  $\gamma_i$  as in (3.14) and  $x_+ \in \mathcal{X}(H_t)$ , we are interested in the spaces

$$\begin{aligned} &\mathcal{R}_1^{0+1}(x_+; \gamma_0) \\ &\mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0). \end{aligned}$$

These consists of all maps  $u: \Sigma \rightarrow \widehat{M[\sigma]}$  for  $\Sigma \in \mathcal{R}_1^{0+1}$  or  $\Sigma \in \mathcal{R}_1^{1+1}$ , respectively, satisfying (4.12) with  $u(E_i) \subset (\phi^{\tau_E})^*L_i$ ,  $u(\zeta_i) = (\phi^{\tau_i})^*\gamma_i$ , and  $u(\zeta_+) = (\phi^{\tau_+})^*x_+$ .

**Lemma 5.9.** *There is a dense subset  $\mathcal{K}_{reg}^{\mathcal{CO}}(M[\sigma]) \subset \mathcal{K}^{\mathcal{CO}}(M[\sigma])$  such that the following hold for every  $\mathbf{K} \in \mathcal{K}_{reg}^{\mathcal{CO}}(M[\sigma])$ .*

- (1) *For any Lagrangian  $L$ , chord  $\gamma_0 \in \mathcal{X}(L, L)$ , and orbit  $x_+ \in \mathcal{X}(H_t)$ ,  $\mathcal{R}_1^{0+1}(x_+; \gamma_0)$  is a smooth manifold of dimension  $\deg(\gamma_0) - \deg(x_+)$ . Additionally, it is empty unless  $n_\sigma(\gamma_0) \leq n_\sigma(x_+)$ .*
- (2) *If  $\deg(\gamma_0) - \deg(x_+) = 0$ , then  $\mathcal{R}_1^{0+1}(x_+; \gamma_0)$  is compact.*

(3) If  $\deg(\gamma_0) - \deg(x_+) = 1$ , then  $\mathcal{R}_1^{0+1}(x_+; \gamma_0)$  admits a Gromov compactification as a compact topological 1-manifold with boundary, and its boundary is in natural bijection with

$$(5.18) \quad \coprod_{\tilde{x} \in \mathcal{X}(H_t)} (\mathcal{R}_1^{0+1}(\tilde{x}; \gamma_0) \times \mathcal{Q}(x_+; \tilde{x})) \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L, L)} (\mathcal{R}(\tilde{\gamma}; \gamma_0) \times \mathcal{R}_1^{0+1}(x_+; \tilde{\gamma})).$$

(4) For any pair of Lagrangians  $L_0$  and  $L_1$ , any pair of chords  $\gamma_0, \gamma_1 \in \mathcal{X}(L_0, L_1)$ , and any orbit  $x_+ \in \mathcal{X}(H_t)$ ,  $\mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$  is a smooth manifold of dimension

$$\deg(\gamma_0) - \deg(\gamma_1) - \deg(x_+) + 1.$$

It is empty unless  $n_\sigma(\gamma_0) \leq n_\sigma(\gamma_1) + n_\sigma(x_+)$ .

(5) If  $\deg(\gamma_0) - \deg(\gamma_1) - \deg(x_+) = -1$ , then  $\mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$  is compact.

(6) If  $\deg(\gamma_0) - \deg(\gamma_1) - \deg(x_+) = 0$ , then  $\mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$  admits a Gromov compactification as a compact topological 1-manifold with boundary, and its boundary is in natural bijection with

$$(5.19) \quad \begin{aligned} & \coprod_{\tilde{x} \in \mathcal{X}(H_t)} (\mathcal{R}_1^{1+1}(\tilde{x}, \gamma_1; \gamma_0) \times \mathcal{Q}(x_+; \tilde{x})) \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_0, L_1)} (\mathcal{R}_1^{1+1}(x_+, \tilde{\gamma}; \gamma_0) \times \mathcal{R}(\gamma_1; \tilde{\gamma})) \\ & \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_0, L_1)} (\mathcal{R}(\tilde{\gamma}; \gamma_0) \times \mathcal{R}_1^{1+1}(x_+, \gamma_1; \tilde{\gamma})) \\ & \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_1, L_1)} (\mathcal{R}^{2+1}(\tilde{\gamma}, \gamma_1; \gamma_0) \times \mathcal{R}_1^{0+1}(x_+; \tilde{\gamma})) \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_0, L_0)} (\mathcal{R}^{2+1}(\gamma_1, \tilde{\gamma}; \gamma_0) \times \mathcal{R}_1^{0+1}(x_+; \tilde{\gamma})). \end{aligned}$$

□

**Definition 5.10.** Suppose  $u \in \mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$  with  $x_+ \in SC_\sigma^*(M[\sigma])$  and  $n_\sigma(\gamma_0) = n_\sigma(\gamma_1)$ . Then, by positivity of intersections,  $u$  doesn't pass through  $D_\sigma$ . Let  $\Sigma$  be the domain of  $u$ , and let  $e: [0, 1] \rightarrow \Sigma$  be a path with  $e(0) \in E_0$  and  $e(1) = \zeta_+$ . Since  $u$  avoids  $D_\sigma$ , so does  $u \circ e$ , and hence the topological intersection number of  $u \circ e$  with  $\sigma(\hat{F} \times \mathbb{R}_+)$  is well defined and independent of the choice of  $e$ . Let  $n_\sigma^{\mathcal{CO}}(u)$  be this number. The **filtered closed-open moduli space**  $\mathcal{R}_1^{1+1, \text{filt}}(x_+, \gamma_1; \gamma_0)$  is the connected component of  $\mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$  consisting of  $u$  with  $n_\sigma^{\mathcal{CO}}(u) = 0$ .

For  $n_\sigma(\gamma_0) < n_\sigma(\gamma_1)$ , we take  $\mathcal{R}_1^{1+1, \text{filt}}(x_+, \gamma_1; \gamma_0)$  to be empty, though one could just as well take it to be all of  $\mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$ .

For  $x \in SC_\sigma^*(M[\sigma])$ , define  $\mathcal{CO}_x^{\text{filt}}: \text{hom}_{\sigma'}^*(L_0, L_1) \rightarrow \text{hom}_{\sigma'}^{*+\deg(x)-1}(L_0, L_1)$  to depend linearly on  $x$  and, for  $x \in \mathcal{X}(H_t)$  a generator, to satisfy

$$(5.20) \quad \begin{aligned} \mathcal{CO}_x^{\text{filt}}(\gamma^m \otimes \cdots \otimes \gamma^0) &= \sum_{\substack{0 \leq i \leq m+1 \\ n_\sigma(\gamma^r) = 0 \forall r < i \\ \tilde{\gamma} \text{ making the result composable} \\ \deg(\tilde{\gamma}) = \deg(x)}} \# \mathcal{R}_1^{0+1}(x; \tilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^i \otimes \tilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0 \\ &+ \sum_{\substack{0 \leq i \leq m \\ n_\sigma(\gamma^r) = 0 \forall r < i \\ \tilde{\gamma} \text{ making the result composable} \\ \deg(\tilde{\gamma}) = \deg(x) + \deg(\gamma^i) - 1}} \# \mathcal{R}_1^{1+1, \text{filt}}(x, \gamma^i; \tilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^{i+1} \otimes \tilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0. \end{aligned}$$

*Remark 5.11.* As with the main homotopy, this is in fact poorly defined in general. However, by choosing a sequence of Floer data  $\mathbf{K}_m$  with collapsing energies, we can again ensure that the problematic terms don't appear when  $m$  is large and  $x$  is a small-action chain supported on the central fiber. In this case, energy alone doesn't exclude the bad components of  $\mathcal{R}_1^{0+1}(x, \tilde{\gamma})$ , but the analog of Lemma 4.25 implies that  $\tilde{\gamma}$  lives in the central fiber. This is impossible for  $L_0$  and  $L_1$ , since they are assumed to be interior Lagrangians of  $M$  and not just  $M[\sigma]$ .

For subsequent moduli spaces, we will implicitly choose Floer data for each  $\mathbf{m}$  and retroactively increase  $\mathbf{m}$  to exclude the new bad terms.

Note, as with the coproduct, that we have only allowed configurations where all long chords occur after the new one. The filtered moduli space is the object which captures the contributions of  $\Delta_y^0$  for which  $n_\sigma(\gamma_a) = 0$  while  $n_\sigma(\gamma_b) > 0$ .

**5.5. Annuli, part 1.** To relate  $A_y^{0,0} + A_y^{0,1}$  with  $\mathcal{CO}_{\mathcal{OC}(y)}^{filt}$ , we follow Abouzaid's construction in [2]. Specifically, we will coherently extend his first and second homotopies to allow for one outer input and verify that the result is a homotopy on  $\text{hom}_{\sigma'}^*(L_0, L_1)$ . This section constructs the first homotopy. The result is a homotopy  $h_y^1$  such that, for  $\delta y = 0$ , the operation  $A_y^{0,0} + A_y^{0,1} + h_y^1 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 h_y^1$  counts analytically gluable broken annuli.

For that, let

$$\mathcal{P}_d^0 = \coprod_{\substack{k,l \geq 0 \\ m \geq 2 \\ k+l+m=d+1}} [0, 1] \times \overline{\mathcal{R}}^{m+1} \times \overline{\mathcal{R}}^{0;k,l}.$$

Note that this differs slightly from Abouzaid's terminology. First, the superscript of zero means that there are no outer inputs, meaning that we should think of gluing the first input  $\zeta_1$  of the  $A_\infty$  disk to the first output  $\zeta_a$  of the coproduct, and likewise we should glue the last input  $\zeta_m$  of the  $A_\infty$  disk to the last output  $\zeta_b$  of the coproduct. Second, we do not bother identifying paired boundary components. When we consider holomorphic curves with domains in  $\mathcal{P}_d^0$ , this means that there will be extra boundary terms which cancel in pairs.

Similarly, let

$$\begin{aligned} \mathcal{P}_d^1 = & \coprod_{\substack{k,l \geq 0 \\ m \geq 2 \\ k+l+m=d+1}} [0, 1] \times \overline{\mathcal{R}}^{m+1} \times \overline{\mathcal{R}}^{1;k,l} \\ & \amalg \coprod_{\substack{k,l \geq 0 \\ m \geq 3 \\ k+l+m=d+2}} [0, 1] \times \overline{\mathcal{R}}^{m+1,1} \times \overline{\mathcal{R}}^{0;k,l} \amalg \coprod_{\substack{k,l \geq 0 \\ m \geq 3 \\ k+l+m=d+2}} [0, 1] \times \overline{\mathcal{R}}^{m+1,m} \times \overline{\mathcal{R}}^{0;k,l}. \end{aligned}$$

Here, the first term is as before except with an outer input in the coproduct disk. For the other two, we have an extra distinguished input on the  $A_\infty$  disk. In this case, we attach  $\zeta_a$  to the first nondistinguished input and  $\zeta_b$  to the last nondistinguished input.

**Definition 5.12.** Before we can start to choose Floer data, we need some auxiliary definitions. For any disk

$$\Sigma \in \overline{\mathcal{R}}^{m+1} \amalg \overline{\mathcal{R}}^{m+1,1} \amalg \overline{\mathcal{R}}^{m+1,m},$$

let

$$\begin{aligned} \zeta_{first} &= \begin{cases} \zeta_1 & \text{for } \Sigma \in \overline{\mathcal{R}}^{m+1} \amalg \overline{\mathcal{R}}^{m+1,m} \\ \zeta_2 & \text{for } \Sigma \in \overline{\mathcal{R}}^{m+1,1} \end{cases} \\ \zeta_{last} &= \begin{cases} \zeta_m & \text{for } \Sigma \in \overline{\mathcal{R}}^{m+1} \amalg \overline{\mathcal{R}}^{m+1,1} \\ \zeta_{m-1} & \text{for } \Sigma \in \overline{\mathcal{R}}^{m+1,m}. \end{cases} \end{aligned}$$

For a two-component stable disk

$$\Sigma \in \partial \left( \overline{\mathcal{R}}^{m+1} \amalg \overline{\mathcal{R}}^{m+1,1} \amalg \overline{\mathcal{R}}^{m+1,m} \right),$$

the **main component** of  $\Sigma$  is the component which contains at least two of  $\{\zeta_0, \zeta_{first}, \zeta_{last}\}$ . For a stable disk with more than two components, the main component is the one for which  $\zeta_0$ ,  $\zeta_{first}$ , and  $\zeta_{last}$  all lie in different directions.

**Definition 5.13.** For  $p = (t, \Sigma^{m+1}, \Sigma^{j;k,l}) \in \bar{\mathcal{P}}_d^j$ , a Lagrangian labeling of  $p$  consists of a Lagrangian labeling for each of  $\Sigma^{m+1}$  and  $\Sigma^{j;k,l}$  such that the labels at  $\zeta_{first} \in \Sigma^{m+1}$  agree with the labels at  $\zeta_a \in \Sigma^{j;k,l}$ , and similarly with  $\zeta_{last}$  and  $\zeta_b$ . A **universal and conformally consistent** choice of Floer data  $\mathbf{K}^{\mathcal{P}}$  for the first homotopy consists, for all  $d \geq 1$  and  $i \in \{0, 1\}$  and each  $p = (t, \Sigma^{m+1}, \Sigma^{j;k,l}) \in \bar{\mathcal{P}}_d^j$  with Lagrangian labels, of a Floer datum  $\mathbf{K}^{\mathcal{P}}(p)$  on  $\Sigma^{m+1}$  with the corresponding labels, such that  $\mathbf{K}^{\mathcal{P}}$  varies smoothly on  $\bar{\mathcal{P}}_d^j$  and has the following properties.

- (1) For  $t = 0$ ,  $\mathbf{K}^{\mathcal{P}}(p)$  agrees up to conformal rescaling with the Floer datum on  $\Sigma^{m+1}$  chosen for the  $A_\infty$  structure.
- (2) For  $t = 1$ , the configuration is glueable to an annulus after a conformal rescaling. Concretely, let  $r_\Delta: \bar{\mathcal{R}}^{j;k,l} \rightarrow (0, \infty)$  be the unique smooth function with  $r_\Delta(\Sigma) = \frac{\tau_b}{\tau_a}$  for  $\Sigma \in \bar{\mathcal{R}}^{j;k,l}$ . Similarly, let

$$r_\mu: \coprod \bar{\mathcal{P}}_d^j \rightarrow (0, \infty)$$

be the unique smooth function with

$$r_\mu(p) = \frac{\tau_{last}(p)}{\tau_{first}(p)},$$

where  $\tau_{first}$  and  $\tau_{last}$  are the rescaling factors that  $\mathbf{K}^{\mathcal{P}}$  assigns to the ends  $\zeta_{first} \in \Sigma^{m+1}$  and  $\zeta_{last} \in \Sigma^{m+1}$ , respectively. We require that

$$r_\mu(1, \Sigma^{m+1}, \Sigma^{j;k,l}) = r_\Delta(\Sigma^{j;k,l}).$$

- (3) If  $\Sigma^{m+1}$  is a nontrivial stable disk, then on every component of  $\Sigma^{m+1}$  aside from the main component,  $\mathbf{K}^{\mathcal{P}}(p)$  is conformally equivalent to the Floer datum chosen for that disk as an element of the associahedron.
- (4) If  $\Sigma^{m+1}$  is a nontrivial stable disk, let  $\Sigma_{main}$  be its main component. If  $\Sigma_{main}$  doesn't contain  $\zeta_{first}$ , let  $\Sigma_{first}$  be the possibly-nodal connected piece of  $\Sigma^{m+1} \setminus \Sigma_{main}$  containing  $\zeta_{first}$ . Likewise, if  $\Sigma_{main}$  doesn't contain  $\zeta_{last}$ , let  $\Sigma_{last}$  be the possibly-nodal connected piece of  $\Sigma^{m+1} \setminus \Sigma_{main}$  containing  $\zeta_{last}$ . Define a probably-nodal disk

$$\Sigma_{big} := \left( \Sigma_{first} \amalg \Sigma^{j;k,l} \amalg \Sigma_{last} \right) / (\zeta_a = \zeta_{first}, \zeta_b = \zeta_{last}) \in \bar{\mathcal{R}}^{j';k',l'}.$$

Then the restriction of  $\mathbf{K}^{\mathcal{P}}(t, \Sigma^{m+1}, \Sigma^{j;k,l})$  to  $\Sigma_{main}$  is conformally equivalent to the Floer datum  $\mathbf{K}^{\mathcal{P}}(t, \Sigma_{main}, \Sigma_{big})$ .

- (5) Suppose  $\Sigma^{j;k,l}$  is a nontrivial stable disk, and that  $\Sigma_{leaf} \subset \Sigma^{j;k,l}$  is an irreducible  $A_\infty$ -type component which is only attached to the rest of  $\Sigma^{j;k,l}$  at the negative puncture. In other words,  $\Sigma_{leaf}$  has one nodal negative puncture, zero other negative punctures, and its positive punctures are all honest positive punctures of  $\Sigma^{j;k,l}$  instead of nodes. Define

$$\Sigma_{small} = \Sigma^{j;k,l} \setminus \Sigma_{leaf}.$$

Then  $\mathbf{K}^{\mathcal{P}}(t, \Sigma^{m+1}, \Sigma^{j;k,l})$  is conformally equivalent to  $\mathbf{K}^{\mathcal{P}}(t, \Sigma^{m+1}, \Sigma_{small})$ .

Denote by  $\mathcal{K}^{\mathcal{P}}(M[\sigma])$  the space of universal and conformally consistent choices of Floer data for the first homotopy.

Suppose we have picked some universal choice  $\mathbf{K}^{\mathcal{P}} \subset \mathcal{K}^{\mathcal{P}}(M[\sigma])$ . For a generator

$$v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_\sigma(\sigma)),$$

let  $B_i \in \mathcal{B}_\sigma(\sigma)$  be such that  $\gamma_i \in \mathcal{X}(B_{i-1}, B_i)$ . For  $L_0, \dots, L_{\bar{j}}$  interior Lagrangians in  $M[\Sigma]$ ,  $\gamma_0 \in \mathcal{X}(L_0, L_{\bar{j}})$ , and

$$\gamma^* = \begin{cases} \text{nothing} & \text{for } \bar{j} = 0 \\ \gamma^1 \in \mathcal{X}(L_0, L_1) & \text{for } \bar{j} = 1, \end{cases}$$

define

$$\mathcal{P}_d^{\bar{j}}(v, \gamma^*; \gamma_0) = \coprod_{\substack{q_a, j, q_b, k, l \geq 0 \\ q_a + j + q_b = \bar{j} \\ k + l < d}} \mathcal{P}_d^{q_a, j, q_b; k, l}(v, \gamma^*; \gamma_0)$$

for certain spaces  $\mathcal{P}_d^{q_a, j, q_b; k, l}(v, \gamma^*; \gamma_0)$ . These describe broken annuli with  $q_a$  outer inputs between the output and the  $a$ -node,  $j$  outer inputs on the coproduct, and  $q_b$  outer inputs between the  $b$ -node and the output. Concretely, this is the union over all

$$p = (t, \Sigma_\mu, \Sigma_\Delta) \in [0, 1] \times \mathcal{R}^{(d-k-l+q_a+q_b+1)+1} \times \mathcal{R}^{j; k, l}$$

of the space of all maps

$$u: \Sigma_\Delta \amalg \Sigma_\mu \rightarrow \widehat{M[\sigma]}$$

satisfying the following conditions.

- (1) Write  $u_\Delta := u|_{\Sigma_\Delta}$  and  $u_\mu := u|_{\Sigma_\mu}$ . Then

$$u_\Delta \in \mathcal{R}^{j; k, l}((\gamma^*), (\gamma_l, \dots, \gamma_{d-k}); \gamma_b, \gamma_a)$$

for some  $\gamma_a \in \mathcal{X}(L_{q_a}, B_l)$  and  $\gamma_b \in \mathcal{X}(B_{d-k-1}, L_{q_a+j})$ .

- (2)  $u_\mu$  satisfies (3.15) for the Floer datum  $\mathbf{K}^{\mathcal{P}}(p)$ .

- (3) Let  $\tau_E(p)$  be the boundary rescaling function assigned to  $\Sigma_\mu$  by  $\mathbf{K}^{\mathcal{P}}(p)$ . Then

$$u(\partial_i \Sigma_\mu) \in \begin{cases} (\phi^{\tau_E(p)})^* L_i & \text{for } i \leq q_a \\ (\phi^{\tau_E(p)})^* L_{\bar{j}+(d-k-l+q_a+q_b+1)-i} & \text{for } i \geq d-k-l+q_a+1 \\ (\phi^{\tau_E(p)})^* B_{l+i-q_a-1} & \text{otherwise} \end{cases}$$

where  $\partial_i \Sigma_\mu$  is the portion of the boundary between  $\zeta_i$  and  $\zeta_{i+1}$ , ordered cyclically.

- (4) Let  $\tau_i$  be the rescaling factor assigned by  $\mathbf{K}^{\mathcal{P}}(p)$  to  $\zeta_i \in \Sigma_\mu$ . Then

$$\begin{aligned} u(\zeta_0) &= (\phi^{\tau_0})^* \gamma_0 \\ u(\zeta_{1+q_a}) &= (\phi^{\tau_{1+q_a}})^* \gamma_a \\ u(\zeta_i) &= (\phi^{\tau_i})^* \gamma_{l+i-q_a-1} \quad \text{for } 2+q_a \leq i \leq d-k-l+q_a \\ u(\zeta_{d-k-l+q_a+1}) &= (\phi^{\tau_{d-k-l+q_a+1}})^* \gamma_b. \end{aligned}$$

This is exhaustive for  $\bar{j} = 0$  or  $\bar{j} = j = 1$ . For  $\bar{j} = 1$  but  $j = 0$ , there is one remaining end. In this case, if  $q_a = 1$ , then we require  $u(\zeta_1) = (\phi^{\tau_1})^* \gamma^*$ . Similarly, if  $q_b = 1$ , we require  $u(\zeta_{d-k-l+2}) = (\phi^{\tau_{d-k-l+2}})^* \gamma^*$ .

For any fixed  $v \in CC_*(\mathcal{B}_\sigma(\sigma))$ , because of action, there are only finitely many choices of intermediate chords  $(\gamma_a, \gamma_b)$  which  $u_\Delta$  can approach. The maximum principle (Lemma A.8) and the usual Gromov compactness argument then imply that the spaces  $\mathcal{P}_d^{q_a, j, q_b; k, l}(v, \gamma^*; \gamma_0)$  have Gromov compactifications  $\overline{\mathcal{P}}_d^{q_a, j, q_b; k, l}(v, \gamma^*; \gamma_0)$  obtained by allowing either or both of  $u_\Delta$  and  $u_\mu$  to break.

**Lemma 5.14.** *There is a dense subset  $\mathcal{K}_{reg}^{\mathcal{P}}(M[\sigma]) \subset \mathcal{K}^{\mathcal{P}}(M[\sigma])$  such that, for every universal choice  $\mathbf{K} \in \mathcal{K}_{reg}^{\mathcal{P}}(M[\sigma])$ , the following hold.*

- (1) For convenience of notation, set  $\deg(\gamma^* = n_\sigma(\gamma^* = 0$  whenever  $\gamma^*$  is nothing (i.e. when  $\bar{j} = 0$ ). Then for any  $d \geq 1$  and all  $q_a, j, q_b \geq 0$  with  $q_a + j + q_b = \bar{j} \leq 1$ , any Lagrangians  $L_0, \dots, L_{\bar{j}}$ , any  $v = \gamma_d \otimes \dots \otimes \gamma_1 \in CC_*(\mathcal{B}_\sigma(\sigma))$ , and any chords  $\gamma_0 \in \mathcal{X}(L_0, L_{\bar{j}})$  and  $\gamma^*$  as above,  $\mathcal{P}_d^{q_a, j, q_b; k, l}(v, \gamma^*; \gamma_0)$  is a smooth manifold with boundary of dimension  $\deg(\gamma_0) - \deg(\gamma^*) - \deg(v) + \bar{j} + 1 - n$ . Additionally, it is empty unless  $n_\sigma(\gamma_0) \leq n_\sigma(\gamma^1)_*$ .
- (2) If  $\deg(\gamma_0) - \deg(\gamma^*) - \deg(v) = n - 1 - \bar{j}$ , then  $\mathcal{P}_d^{q_a, j, q_b; k, l}(v, \gamma^*; \gamma_0)$  is compact.
- (3) If  $\deg(\gamma_0) - \deg(\gamma^*) - \deg(v) = n - \bar{j}$ , then  $\overline{\mathcal{P}}_d^{q_a, j, q_b; k, l}(v, \gamma^*; \gamma_0)$  is a compact topological 1-manifold with boundary, and its boundary consists of all broken configurations  $u$  of the following types.

In the first type of configuration,  $t \in (0, 1)$ ,

$$u_\Delta \in \overline{\partial \mathcal{R}^{j; k, l}}((\gamma^?), (\gamma_l, \dots, \gamma_{d-k}); \gamma_b, \gamma_a),$$

where

$$(5.21) \quad \gamma^? = \begin{cases} \text{nothing} & \text{for } j = 0 \\ \gamma^* = \gamma^1 & \text{for } j = 1, \end{cases}$$

while  $u_\mu$  is a map  $u_\mu: \Sigma_\mu \rightarrow \widehat{M}[\sigma]$  satisfying the Cauchy-Riemann equation perturbed by the Floer datum chosen for  $(t, \Sigma_\mu, \Sigma_\Delta)$  with the corresponding boundary and asymptotic conditions.

In the second type of configuration,  $t \in (0, 1)$ ,

$$u_\Delta \in \mathcal{R}^{j; k, l}(\gamma^?, (\gamma_l, \dots, \gamma_{d-k}); \gamma_b, \gamma_a),$$

- (5.22) while  $u_\mu$  has broken, such that one component is an honest  $A_\infty$  disk  $u_\mu^0$  which has neither  $\gamma_a$  nor  $\gamma_b$  as an input. Such a disk is either a portion of the Hochschild differential on  $v$ , or it is a 1- or 2-input disk involving  $\gamma^*$  and/or  $\gamma_0$ . The other disk,  $u_\mu^{\text{main}}$ , is a map  $u_\mu^{\text{main}}: \Sigma_\mu^{\text{main}} \rightarrow \widehat{M}[\sigma]$  satisfying the Cauchy-Riemann equation perturbed by the Floer datum chosen for  $(t, \Sigma_\mu^{\text{main}}, \Sigma_\Delta)$  with the induced boundary and asymptotic conditions.

This configuration is part of some  $\mathcal{P}_{d'}^{\bar{j}'}(v', (\gamma^*)'; \gamma_0)$ , where  $\bar{j}' = \bar{j}$  unless  $\bar{j} = 1$  and  $u_\mu^0$  involves both  $\gamma^1$  and  $\gamma_0$ .

In the third type of configuration,  $t \in (0, 1)$ ,

$$u_\Delta \in \mathcal{R}^{j; k, l}((\gamma^?), (\gamma_l, \dots, \gamma_{d-k}); \gamma_b, \gamma_a),$$

- (5.23) while  $u_\mu$  has broken, such that one component is an honest  $A_\infty$  disk  $u_\mu^{\text{in}}$  which has  $\gamma_a$  or  $\gamma_b$  as an input but does not have  $\gamma_0$  as its output. The other,  $u_\mu^{\text{out}}$ , is a map  $u_\mu^{\text{out}}: \Sigma_\mu^{\text{out}} \rightarrow \widehat{M}[\sigma]$  satisfying the Cauchy-Riemann equation perturbed by the Floer datum chosen for  $(t, \Sigma_\mu^{\text{out}}, \Sigma_{\text{big}})$  with the induced boundary and asymptotic conditions, where  $\Sigma_{\text{big}}$  is the broken disk formed by joining the domains of  $u_\Delta$  and  $u_\mu^{\text{in}}$ .

- (5.24) In the fourth type of configuration,  $t = 0$ , in which case  $u$  is a two-component broken annulus of the type contributing to  $A_v^{0, \bar{j}}$ . Indeed,  $A_v^{0, \bar{j}}$  is a count of precisely such annuli satisfying either (1)  $n_\sigma(\gamma^*) = 0$ , or (2)  $q_a = 0$ ,  $n_\sigma(\gamma_a) = 0$ , and  $n_\sigma(\gamma_0) = n_\sigma(\gamma^1)$ .

(5.25) *In the fifth type of configuration,  $t = 1$ , in which case  $u$  is a two-component broken annulus which can be glued into an honest perturbed holomorphic annulus.*

□

While  $\mathcal{P}_d^0(v; \gamma_0)$  already extends the moduli space giving rise to  $A_v^{0,0}$ , for  $A_v^{0,1}$  we need to look at a connected component of  $P_d^1(v, \gamma^1; \gamma_0)$  as hinted by (5.24). This is the space  $\mathcal{P}_{d, \text{filt}}^1(v, \gamma^1; \gamma_0)$  consisting of those  $u \in P_d^1(v, \gamma^1; \gamma_0)$  for which either (1)  $n_\sigma(\gamma^1) = 0$ , or (2)  $q_a = 0$ ,  $n_\sigma(\gamma_a) = 0$ , and  $n_\sigma(\gamma_0) = n_\sigma(\gamma^1)$ . For an equivalent description closer in spirit to the filtered closed-open moduli space, choose for all  $\Sigma_\mu$  a path  $e: [0, 1] \rightarrow \Sigma_\mu$  starting on the edge  $\partial_0 \Sigma_\mu$  and ending on  $\partial_{1+q_a} \Sigma_\mu$ .  $\mathcal{P}_{d, \text{filt}}^1(v, \gamma^1; \gamma_0)$  is the space of all  $u$  which avoid  $D_\sigma$  and for which the topological intersection number  $u \circ e$  with  $\sigma(\hat{F} \times \mathbb{R}_+)$  vanishes.

Define a linear map  $h_y^1: \text{hom}_{\sigma'}^*(L_0, L_1) \rightarrow \text{hom}_{\sigma'}^{*+\deg(v)+n-2}(L_0, L_1)$  to depend linearly on a low-action stabilization  $y$  and, for  $v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_\sigma(\sigma))$  a generator, to be given by

$$(5.26) \quad \begin{aligned} h_v^1(\gamma^m \otimes \cdots \otimes \gamma^0) &= \sum_{\substack{0 \leq i \leq m+1 \\ n_\sigma(\gamma^r)=0 \forall r < i \\ \tilde{\gamma} \text{ making the result composable} \\ \deg(\tilde{\gamma})=\deg(v)+n-1}} \# \mathcal{P}_d^0(v; \tilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^i \otimes \tilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0 \\ &+ \sum_{\substack{0 \leq i \leq m \\ n_\sigma(\gamma^r)=0 \forall r < i \\ \tilde{\gamma} \text{ making the result composable} \\ \deg(\tilde{\gamma})=\deg(v)+\deg(\gamma^i)+n-2}} \# \mathcal{P}_{d, \text{filt}}^1(v, \gamma^i; \tilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^{i+1} \otimes \tilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0. \end{aligned}$$

In the same way, we define the **gluable annulus maps**

$$\tilde{A}_y^{0,0} \text{ and } \tilde{A}_y^{0,1}: \text{hom}_{\sigma'}^*(L_0, L_1) \rightarrow \text{hom}_{\sigma'}^{*+\deg(v)+n-1}(L_0, L_1)$$

via

$$(5.27a) \quad \tilde{A}_v^{0,0}(\gamma^m \otimes \cdots \otimes \gamma^0) = \sum_{\substack{0 \leq i \leq m+1 \\ n_\sigma(\gamma^r)=0 \forall r < i \\ \tilde{\gamma} \text{ making the result composable} \\ \deg(\tilde{\gamma})=\deg(v)+n}} \# [\mathcal{P}_d^0(v; \tilde{\gamma})]_{t=1} \cdot \gamma^m \otimes \cdots \otimes \gamma^i \otimes \tilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0$$

$$(5.27b) \quad \tilde{A}_v^{0,1}(\gamma^m \otimes \cdots \otimes \gamma^0) = \sum_{\substack{0 \leq i \leq m \\ n_\sigma(\gamma^r)=0 \forall r < i \\ \tilde{\gamma} \text{ making the result composable} \\ \deg(\tilde{\gamma})=\deg(v)+\deg(\gamma^i)+n-1}} \# [\mathcal{P}_{d, \text{filt}}^1(v, \gamma^i; \tilde{\gamma})]_{t=1} \cdot \gamma^m \otimes \cdots \otimes \gamma^{i+1} \otimes \tilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0,$$

where the notation  $[\cdot]_{t=1}$  refers to the portion of the corresponding moduli space which occurs at  $t = 1$ . This is the portion of the boundary of the 1-dimensional part of the filtered moduli space described in (5.25).

**Lemma 5.15.** *Up to terms which decrease the main filtration,*

$$(5.28) \quad h_v^1 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 h_v^1 = h_{\delta_v}^1 + A_v^{0,0} + A_v^{0,1} + \tilde{A}_v^{0,0} + \tilde{A}_v^{0,1}.$$

*Proof.* Write  $h_v^1 = (h_v^1)^{0,0} + (h_v^1)^{0,1}$ , where  $(h_v^1)^{0,0}$  is the part of  $h_v^1$  coming from the first sum in (5.26), while  $(h_v^1)^{0,1}$  is the part coming from the second sum. We begin by analyzing the part of  $h_v^1 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 h_v^1$  which increases word length. This consists of all ways of applying  $(h_v^1)^{0,0}$  and  $\mu_{\mathcal{W}_{\sigma'}(M[\sigma])}^1$  in some order. All such terms cancel except those in which  $\mu_{\mathcal{W}_{\sigma'}(M[\sigma])}^1$  is applied to the output of  $h_v^1$ , which we see constitute the part of (5.22) for  $\bar{j} = 0$  which do not contribute to

the Hochschild differential. We therefore examine the rest of the boundary of the corresponding 1-dimensional moduli space.

The portion of the Hochschild differential not in (5.22) appears in (5.21), giving rise to a  $(h_{\delta v}^1)^{0,0}$ . The rest of (5.21) for  $\bar{j} = 0$  comes from the breaking of an  $A_\infty$  disk outputting  $\zeta_a$  or  $\zeta_b$ . Such disks precisely form the contribution of (5.23) for a different connected component, and so they cancel in pairs. The remaining terms (5.24) and (5.25) correspond precisely to  $A_v^{0,0}$  and  $\tilde{A}_v^{0,0}$ , respectively, which confirms the portion of (5.28) which increases word length.

For the portion which preserves word length, in order to avoid leaving the filtered moduli space, we consider only the part of  $\mu_{\sigma'}^1$ , which does not decrease intersection number. Among such terms, we are interested in all ways of performing both  $\mu_{\mathcal{W}_{\sigma'}(M[\sigma])}^2$  and  $(h_v^1)^{0,0}$  or both  $\mu_{\mathcal{W}_{\sigma'}(M[\sigma])}^1$  and  $(h_v^1)^{0,1}$  in some order. These again cancel when the operations take place at different places in  $\gamma^m \otimes \cdots \otimes \gamma^0 \in \text{hom}_{\sigma'}(L_0, L_1)$ . The remaining terms include not just the non-Hochschild part of (5.22) for  $\bar{j} = 1$ , but also the component of (5.21) given by a Floer strip escaping at the outer input of the coproduct, as in (5.5f). The rest of the argument proceeds as above.  $\square$

**5.6. Annuli, part 2.** We can now construct the homotopy between  $\tilde{A}_y^{0,0} + \tilde{A}_y^{0,1}$  and  $\mathcal{CO}_{\mathcal{OC}(y)}^{filt}$  for a Hochschild cycle  $y \in CC_*(\mathcal{B}_\sigma(\sigma))$ . This is the induced effect of the Cardy relation for wrapped Floer theory [2, 19] on the quotient category, modified to interact with the intersection filtration.

For  $d \geq 1$ , let  $\mathcal{A}_d^{0+1}$  be the space of conformal annuli  $\Sigma$  with the following data

- (1)  $d$  punctures on the inner boundary, labeled  $\zeta_1$  through  $\zeta_d$  in clockwise order. Note that this becomes a standard counterclockwise ordering after exchanging the inner and outer boundary components.
- (2) One puncture  $\zeta_0$  on the outer boundary component, such that in coordinates

$$(5.29) \quad \text{int}(\Sigma) = \{z \in \mathbb{C} \mid 1 < |z| < R\}$$

for some  $R > 1$  with  $\zeta_d = 1$ , we have  $\zeta_0 = -R$ .

$\mathcal{A}_d^{0+1}$  admits a Deligne-Mumford compactification  $\overline{\mathcal{A}}_d^{0+1}$  which is a manifold with stratified boundary. In addition to ordinary smooth corners, it has boundary strata of codimension at least two for which neighborhoods are subvarieties of the standard corner  $[0, a)^i \times (-a, a)^j$ . This is similar to the situation with multiplihedra. Its codimension 1 boundary components come in three types.

- (1) The first type occurs as some of the inner boundary punctures come together while  $R$  remains finite and strictly greater than 1. Such configurations are described by

$$(5.30) \quad \coprod_{\substack{2 \leq k \leq d \\ 1 \leq i \leq k}} \mathcal{A}_{d+1-k}^{0+1} \times \mathcal{R}^{k+1,i} \quad \amalg \quad \coprod_{\substack{2 \leq k \leq d-1 \\ 1 \leq i \leq d-k}} \mathcal{A}_{d+1-k}^{0+1} \times \mathcal{R}^{k+1}$$

as in (4.14). As before, the index  $i$  in the second term keeps track of where the punctures collided.

- (2) The second type occurs as  $R$  tends to 1. Because of the anti-alignment condition on  $\zeta_0$  and  $\zeta_d$ ,  $\Sigma$  has to break into a nodal configuration in which  $\zeta_0$  and  $\zeta_d$  are on different irreducible components. Thus,  $\Sigma$  must have at least two components, and the codimension 1 condition is that it breaks into exactly two components. Such configurations are described by

$$\coprod_{\substack{k, l \geq 0 \\ m \geq 2 \\ k+l+m=d+1}} \mathcal{R}^{m+1} \times \mathcal{R}^{0;k,l},$$

which we identify with  $\text{int}[\mathcal{P}_d^0]_{t=1}$ , the interior of the portion of  $\mathcal{P}_d^0$  lying over  $t = 1$ .



- (3) The third type occurs as  $R$  tends to  $\infty$ . In this case, we obtain two disks attached nodally at their centers, and the anti-alignment condition gives rise to a preferred angular gluing parameter. The configuration is thus parametrized by

$$\mathcal{R}_1^{0+1} \times \mathcal{R}_d^1,$$

where the alignment conditions on the cylindrical ends for  $\mathcal{CO}$  and  $\mathcal{OC}$  implements the restriction on gluing angles. We then obtain boundary charts by introducing a gluing parameter  $\rho$  satisfying  $\rho = \frac{1}{\log \ell}$ , where  $\ell$  is the gluing length for the cylindrical ends.

The higher codimension strata are either combinations of the above or paired boundary strata of  $[\mathcal{P}_d^0]_{t=1}$ , or in other words configurations which arise as the boundary of two different components of  $[\mathcal{P}_d^0]_{t=1}$ .

Next, let  $\mathcal{A}_d^{1+1}$  for  $d \geq 1$  be the space of conformal annuli  $\Sigma$  with the following data

- (1)  $d$  punctures on the inner boundary, labeled  $\zeta_1$  through  $\zeta_d$  in clockwise order. Note that this becomes a standard counterclockwise ordering after exchanging the inner and outer boundary components.
- (2) Two punctures  $\zeta_0$  and  $\zeta^1$  on the outer boundary component, such that in coordinates (5.29) with  $\zeta_d = 1$ , we have  $\zeta_0 = -R$ .

$\mathcal{A}_d^{1+1}$  admits a Deligne-Mumford compactification  $\overline{\mathcal{A}}_d^{1+1}$  which is again a manifold with stratified boundary. Its codimension 1 boundary components come in four types, the first three of which are essentially the same as for  $\overline{\mathcal{A}}_d^{0+1}$ .

- (1) The first type occurs as some of the inner boundary punctures come together while  $R$  remains finite and strictly greater than 1. Such configurations are described by

$$(5.31) \quad \coprod_{\substack{2 \leq k \leq d \\ 1 \leq i \leq k}} \mathcal{A}_{d+1-k}^{1+1} \times \mathcal{R}^{k+1,i} \quad \amalg \quad \coprod_{\substack{2 \leq k \leq d-1 \\ 1 \leq i \leq d-k}} \mathcal{A}_{d+1-k}^{1+1} \times \mathcal{R}^{k+1}.$$

- (2) The second type occurs as  $R$  tends to 1. Such configurations can be identified with  $\text{int}[\mathcal{P}_d^1]_{t=1}$ , the interior of the portion of  $\mathcal{P}_d^1$  lying over  $t = 1$ .
- (3) The third type occurs as  $R$  tends to  $\infty$  and is parametrized by

$$\mathcal{R}_1^{1+1} \times \mathcal{R}_d^1.$$

- (4) The fourth type occurs when  $\zeta^1$  collides with  $\zeta_0$  while  $R \in (1, \infty)$ . This case is formally similar to the first, but here we reduce the number of outer punctures. The configurations are described by

$$\mathcal{R}^{2+1} \times \mathcal{A}_d^{0+1}.$$

As before, the higher codimension strata are either combinations of the above or paired boundary strata of  $[\mathcal{P}_d^1]_{t=1}$ .

A collection of strip-like ends for an annulus  $\Sigma \in \mathcal{A}_d^{j+1}$  consists of positive strip-like ends  $\epsilon_i$  at  $\zeta_i$  for  $i \in \{1, \dots, d\}$  and, if applicable,  $\epsilon^1$  at  $\zeta^1$ , along with a negative strip-like end  $\epsilon_0$  at  $\zeta_0$ , such that the images of the ends are pairwise disjoint. A cylinder for  $\Sigma$  is a finite cylinder  $\delta: [a, b] \times S^1 \rightarrow \Sigma$  which is disjoint from the strip-like ends and, in the coordinates (5.29) with  $\zeta_d = 1$  and  $\zeta_0 = -R$ , takes the form

$$\delta(s, t) = ce^{-2\pi(s+it)} \quad \text{with } c \in \mathbb{R} \text{ positive.}$$

A **universal choice of ends and cylinders** for  $\mathcal{A}_d^{j+1}$  consists, for all  $d \geq 1$  and  $j \in \{0, 1\}$ , of a collection of strip-like ends for each  $\Sigma \in \mathcal{A}_d^{j+1}$  which varies smoothly over  $\mathcal{A}_d^{j+1}$ , along with a cylinder for  $\Sigma$  whenever  $R \geq 2$  which also varies smoothly over  $\mathcal{A}_d^{j+1}$ , which satisfy

- (1) The strip-like ends agree to infinite order at the boundary with the collection of strip-like ends induced by gluing.

- (2) Near  $R = \infty$ , the cylinder agrees with the finite cylinder induced by gluing.
- (3) When  $R = 2$ , the width  $b - a$  of the cylinder is zero.

Fix once and for all a universal choice of strip-like ends and cylinders for  $\mathcal{A}_d^{j+1}$ .

Similarly, a **universal and conformally consistent** choice of Floer data for  $\mathcal{A}_d^{j+1}$  consists, for all  $d \geq 0$  and  $j \in \{0, 1\}$ , of a Floer datum for each  $\Sigma \in \mathcal{A}_d^{j+1}$  varying smoothly over  $\mathcal{A}_d^{j+1}$ , and such that at the boundary it agrees to infinite order with the conformal class of Floer data induced by gluing. It is easy to see that conformal consistency can be achieved, at least away from the  $R = 1$  boundary of  $\overline{\mathcal{A}}_d^{j+1}$ . At the  $R = 1$  boundary, one needs the observation that, for fixed  $d$ , the Floer data on paired boundary strata of  $\mathcal{P}_d^j$  agree up to a global conformal factor, so consistency can be extended across the corresponding strata of  $\partial \overline{\mathcal{A}}_d^{j+1}$ . Let  $\mathcal{K}^{\mathcal{A}}(M[\sigma])$  denote the space of all universal and conformally consistent choices of Floer data for  $\mathcal{A}_d^{j+1}$ .

Given  $\mathbf{K}^{\mathcal{A}} \in \mathcal{K}^{\mathcal{A}}(M[\sigma])$ , we obtain spaces of holomorphic annuli. For  $j = 0$ , these are specified by a generator  $v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_{\sigma}(\sigma))$  and a chord  $\gamma_0 \in \mathcal{X}(L, L)$  with  $L$  an interior Lagrangian in  $M[\sigma]$ . The resulting moduli space

$$\mathcal{A}_d^{0+1}(v; \gamma_0)$$

is the space of all maps  $u: \Sigma \rightarrow \widehat{M[\sigma]}$  for  $\Sigma \in \mathcal{A}_d^{0+1}$  satisfying (4.12) such that  $u(\zeta_i) = (\phi^{\tau_i})^* \gamma_i$  and with the corresponding boundary conditions. Similarly, if  $L_0$  and  $L_1$  are interior Lagrangians of  $M[\sigma]$  and  $\gamma_0, \gamma^1 \in \mathcal{X}(L_0, L_1)$ , then we obtain

$$\mathcal{A}_d^{1+1}(v, \gamma^1; \gamma_0),$$

the space of perturbed holomorphic curves  $u$  with domain in  $\mathcal{A}_d^{1+1}$  such that  $u(\zeta_i) = (\phi^{\tau_i})^* \gamma_i$  and  $u(\zeta^1) = (\phi^{\tau^1})^* \gamma^1$ , and which satisfy the appropriate boundary conditions.

As usual,  $\mathcal{A}_d^{i+1}(v; \gamma^i, \gamma_0)$  have Gromov compactifications  $\overline{\mathcal{A}}_d^{i+1}(v; \gamma^i, \gamma_0)$  obtained by including broken configurations.

**Lemma 5.16.** *There is a dense subset  $\mathcal{K}_{reg}^{\mathcal{A}}(M[\sigma]) \subset \mathcal{K}^{\mathcal{A}}(M[\sigma])$  such that, for every universal choice  $\mathbf{K}^{\mathcal{A}} \in \mathcal{K}_{reg}^{\mathcal{A}}(M[\sigma])$ , the following hold.*

- (1) *For any  $d \geq 1$ , any Lagrangian  $L$ , any generator  $v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_{\sigma}(\sigma))$ , and any chord  $\gamma_0 \in \mathcal{X}(L, L)$ ,  $\mathcal{A}_d^{0+1}(v; \gamma_0)$  is a smooth manifold of dimension*

$$\deg(\gamma_0) - \deg(v) + 1 - n.$$

*Additionally, it is empty unless  $n_{\sigma}(\gamma_0) = 0$ .*

- (2) *If  $\deg(\gamma_0) - \deg(v) = n - 1$ , then  $\mathcal{A}_1^{0+1}(v; \gamma_0)$  is compact.*
- (3) *If  $\deg(\gamma_0) - \deg(v) = n$ , then  $\overline{\mathcal{A}}_1^{0+1}(v; \gamma_0)$  is a compact topological 1-manifold with boundary, and its boundary consists of all broken configurations of the following types.*

(5.32) *The first type corresponds to the domain hitting a boundary stratum of  $\overline{\mathcal{A}}_d^{0+1}$  of the form (5.30) or a Floer strip breaking off at a puncture on the inner boundary. In symbols, these are essentially the same as the first four terms of (4.16), though there we separated the chords making up  $v$ .*

(5.33) *The second type comes from a Floer strip breaking off at  $\gamma_0$  and is parametrized by*

$$\coprod_{\tilde{\gamma} \in \mathcal{X}(L, L)} \mathcal{R}(\tilde{\gamma}; \gamma_0) \times \mathcal{A}_d^{0+1}(v; \tilde{\gamma}).$$

(5.34) *The third type comes from the domain hitting the  $R = 1$  boundary and is precisely  $[\mathcal{P}_d^0(v; \gamma_0)]_{t=1}$ .*

(5.35) *The fourth type comes from the domain hitting  $R = \infty$  and is parametrized by*

$$\coprod_{\tilde{x} \in \mathcal{X}(H_t)} \mathcal{R}_1^{0+1}(\tilde{x}; \gamma_0) \times \mathcal{R}_d^1(\gamma_d, \dots, \gamma_1; \tilde{x})$$

(4) *For any  $d \geq 1$  any Lagrangians  $L_0, L_1$ , any generator  $v = \gamma_d \otimes \dots \otimes \gamma_1 \in CC_*(\mathcal{B}_\sigma(\sigma))$ , and any chords  $\gamma_0, \gamma^1 \in \mathcal{X}(L_0, L_1)$ ,  $\mathcal{A}_1^{1+1}(v, \gamma^1; \gamma_0)$  is a smooth manifold of dimension  $\deg(\gamma_0) - \deg(\gamma^1) - \deg(v) + 2 - n$ . Additionally, it is empty unless  $n_\sigma(\gamma_0) \leq n_\sigma(\gamma^1)$ .*

(5) *If  $\deg(\gamma_0) - \deg(\gamma^1) - \deg(v) = n - 2$ , then  $\mathcal{A}_1^{1+1}(v, \gamma^1; \gamma_0)$  is compact.*

(6) *If  $\deg(\gamma_0) - \deg(\gamma^1) - \deg(v) = n - 1$ , then  $\overline{\mathcal{A}}_1^{1+1}(v, \gamma^1; \gamma_0)$  is a compact topological 1-manifold with boundary, and its boundary consists of all broken configurations of the following types.*

(5.36) *The first type corresponds to the domain hitting a boundary stratum of  $\overline{\mathcal{A}}_d^{1+1}$  of the form (5.31) or a Floer strip breaking off at a puncture on the inner boundary. In symbols, these are also essentially the same as the first four terms of (4.16).*

*The second type comes from a Floer strip breaking off at  $\gamma^1$  or  $\gamma_0$  or from a collision of  $\zeta^1$  with  $\zeta_0$ . Such configurations are parametrized by*

$$(5.37) \quad \coprod_{\tilde{\gamma} \in \mathcal{X}(L_0, L_1)} \mathcal{A}_d^{1+1}(v, \tilde{\gamma}; \gamma_0) \times \mathcal{R}(\gamma^1; \tilde{\gamma}) \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_0, L_1)} \mathcal{R}(\tilde{\gamma}; \gamma_0) \times \mathcal{A}_d^{1+1}(v, \gamma^1; \tilde{\gamma})$$

$$\amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_0, L_0)} \mathcal{R}^{2+1}(\gamma^1, \tilde{\gamma}; \gamma_0) \times \mathcal{A}_d^{0+1}(v; \tilde{\gamma}) \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_1, L_1)} \mathcal{R}^{2+1}(\tilde{\gamma}, \gamma^1; \gamma_0) \times \mathcal{A}_d^{0+1}(v; \tilde{\gamma}).$$

(5.38) *The third type comes from the domain hitting the  $R = 1$  boundary and is precisely  $[\mathcal{P}_d^1(v, \gamma^1; \gamma_0)]_{t=1}$ .*

(5.39) *The fourth type comes from the domain hitting  $R = \infty$  and is parametrized by*

$$\coprod_{\tilde{x} \in \mathcal{X}(H_t)} \mathcal{R}_1^{1+1}(\tilde{x}, \gamma^1; \gamma_0) \times \mathcal{R}_d^1(\gamma_d, \dots, \gamma_1; \tilde{x})$$

□

To extend the filtered versions of the moduli spaces for the closed-open maps and the first homotopy, choose for all  $\Sigma \in \mathcal{A}_d^{1+1}$  a path  $e: [0, 1] \rightarrow \Sigma$  such that  $e(0)$  is on the outer boundary component to the right of  $\zeta_0$  and  $e(1)$  is on the inner boundary. Then for any  $\gamma_0$  and  $\gamma^1$  with  $n_\sigma(\gamma_0) = n_\sigma(\gamma^1)$  and any  $u \in \mathcal{A}_d^{1+1}(v; \gamma^1; \gamma_0)$ ,  $u \circ e$  is a path between interior Lagrangians which avoids  $D_\sigma$ , so it has a well defined intersection number with  $\sigma(\hat{F} \times \mathbb{R}_+)$ . Since the chords  $\gamma_i$  for  $i > 0$  have  $n_\sigma(\gamma_i) = 0$ , we can homotope the end of  $e$  through  $\zeta_i$  without changing the intersection number. This implies that the intersection number is independent of the choice of  $e$ , and so we call it  $n_\sigma^{\mathcal{A}}(u)$ . The space

$$\mathcal{A}_{d, \text{filt}}^{1+1}(v, \gamma^+; \gamma_0)$$

consists of all  $u \in \mathcal{A}_d^{1+1}(v; \gamma^1; \gamma_0)$  which avoid  $D_\sigma$  and satisfy  $n_\sigma^{\mathcal{A}}(u) = 0$ .

The space  $\mathcal{A}_{d, \text{filt}}^{1+1}(v, \gamma^+; \gamma_0)$  is a union of connected components of  $\mathcal{A}_d^{1+1}(v, \gamma^+; \gamma_0)$ , and its boundary inherits the filtered condition. In other words, they are the same except in the following two

ways. First, all annuli, broken annuli, and closed-open maps are replaced by their filtered versions. Second, for  $n_\sigma(\gamma_0) = n_\sigma(\gamma^1) > 0$ , the terms

$$\coprod_{\tilde{\gamma} \in \mathcal{X}(L_1, L_1)} \mathcal{R}^{2+1}(\tilde{\gamma}, \gamma^1; \gamma_0) \times \mathcal{A}_d^{0+1}(v; \tilde{\gamma})$$

in (5.37) no longer contribute.

Define a linear map  $h_y^2: \text{hom}_{\sigma'}^*(L_0, L_1) \rightarrow \text{hom}_{\sigma'}^{*+\deg(v)+n-2}(L_0, L_1)$  to depend linearly on  $y$  and, for  $v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_\sigma(\sigma))$ , to be given by

$$(5.40) \quad \begin{aligned} h_v^2(\gamma^m \otimes \cdots \otimes \gamma^0) &= \sum_{\substack{0 \leq i \leq m+1 \\ n_\sigma(\gamma^r)=0 \forall r < i \\ \tilde{\gamma} \text{ making the result composable} \\ \deg(\tilde{\gamma})=\deg(v)+n-1}} \# \mathcal{A}_d^{0+1}(v; \tilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^i \otimes \tilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0 \\ &+ \sum_{\substack{0 \leq i \leq m \\ n_\sigma(\gamma^r)=0 \forall r < i \\ \tilde{\gamma} \text{ making the result composable} \\ \deg(\tilde{\gamma})=\deg(v)+\deg(\gamma^i)+n-2}} \# \mathcal{A}_{d, \text{filt}}^{1+1}(v, \gamma^i; \tilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^{i+1} \otimes \tilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0. \end{aligned}$$

By essentially the same argument as for Lemma 5.15, we conclude

**Lemma 5.17.** *Up to terms which decrease the main filtration,*

$$(5.41) \quad h_v^2 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 h_v^2 = h_{\delta v}^2 + \tilde{A}_v^{0,0} + \tilde{A}_v^{0,1} + \mathcal{CO}_{\mathcal{OC}(v)}^{filt}.$$

□

**5.7. The last homotopy.** Our goal now is to construct, for a saddle unit  $f_\sigma \in SC_\sigma^1(M[\sigma])$ , a homotopy  $h_{f_\sigma}^3$  between  $\mathcal{CO}_{f_\sigma}^{filt}$  and an operation  $\text{id}_\sigma$  which, while not the identity, induces the identity on the portion of the associated graded of  $\text{hom}_{\sigma'}^*(L_0, L_1)$  which does not lie in  $\text{hom}_{\sigma'}^*(L_0, L_1)$ .

Thus, let  $K^f$  be a Floer datum on  $\mathbb{C}$  giving rise to  $f_\sigma$  as in 4.20. Denote by  $D^Y$  the closed unit disk with a puncture  $\zeta_0^Y$  at  $-1$ , which we equip with a negative strip-like end  $\epsilon_0^Y$  and a family of finite cylinders  $\delta^Y$  as follows. Let  $D_1^{0,1}$  be as in Section 5.4 with negative strip-like end  $\epsilon_0$ . Then  $\epsilon_0$  induces  $\epsilon_0^Y$  via the unique biholomorphism  $D_1^{0,1} \rightarrow D^Y \setminus \{0\}$ . For the cylinders, we are interested in a  $(0, \frac{1}{2}]$ -parametrized family

$$\delta^{D,Y}(\rho): [a_\rho, b_\rho] \times S^1 \rightarrow \text{int}(D^Y)$$

which satisfies

- (1) For  $\rho$  close to 0,  $\delta^{D,Y}(\rho)$  agrees with the finite cylinder obtained by gluing  $\epsilon^f$  on  $\mathbb{C}$  to  $\epsilon_+$  on  $D_1^{0,1}$  with length  $e^{\frac{1}{\rho}}$ . Here, we are implicitly using the biholomorphism from the glued surface to  $D^Y$  which sends  $0 \in \mathbb{C}$  to  $0 \in D^Y$  and  $\zeta_0$  to  $\zeta_0^Y$ .
- (2)  $b_{\frac{1}{2}} = a_{\frac{1}{2}}$ . In other words, at  $\rho = \frac{1}{2}$ , the cylinder has width zero.

We will think of this data as a  $(0, 1]$ -parametrized space of Riemann surfaces with boundary, ends, and cylinders  $D^Y(\rho)$ , which for  $\rho \in (0, \frac{1}{2}]$  is equipped with the strip-like end  $\epsilon_0^Y$  and finite cylinder  $\delta^{D,Y}(\rho)$  and for  $\rho > \frac{1}{2}$  is equipped only with  $\epsilon_0^Y$ .

We consider smooth families of Floer data  $\mathbf{K}^{D,Y}(\rho)$  on  $D^Y(\rho)$  such that, in a gluing chart near  $\rho = 0$ ,  $\mathbf{K}^{D,Y}$  extends smoothly to 0, where it is conformally equivalent to the Floer data chosen for  $\mathbb{C}$  and  $D_1^{0,1}$ . Let  $\mathcal{K}^{D,Y}(M[\sigma])$  denote the space of such families.

In addition to the above data, choose  $p_D: (0, 1] \rightarrow [0, 1]$  to be a nondecreasing smooth function which is 0 on  $(0, \frac{1}{3}]$  and 1 on  $[\frac{1}{2}, 1]$ . Similarly, choose a smooth isotopy  $Y_\sigma(\rho)$  of properly embedded hypersurfaces in  $\widehat{M[\sigma]}$  which avoid  $\sigma(\hat{F} \times \mathbb{R}_{\geq 0})$  for all  $\rho$  and satisfy the following conditions. First,

$Y_\sigma(\rho) = Y_\sigma$  for  $\rho \leq \frac{1}{4}$ . Second,  $Y_\sigma(\rho) \in \text{image}(\sigma)$  for  $\rho \geq \frac{1}{3}$ , and moreover  $Y_\sigma(1)$  is transverse to all chords between interior Lagrangians which appear in  $\mathcal{W}_{\sigma'}(M[\sigma])$ .

Let  $L$  be an interior Lagrangian of  $M[\sigma]$ , and let  $\gamma \in \mathcal{X}(L, L)$ . Given a family of Floer data  $\mathbf{K}^{D,Y} \in \mathcal{K}^{D,Y}(M[\sigma])$ , let  $\mathcal{D}^Y(\gamma)$  be the union over all  $\rho \in (0, 1]$  of the spaces  $\mathcal{D}_\rho^Y(\gamma)$  of maps

$$u: D^Y \rightarrow \widehat{M[\sigma]}$$

satisfying the following conditions.

- (1)  $u$  satisfies (4.12) for  $\mathbf{K}^{D,Y}(\rho)$ .
- (2)  $u(z) \in (\phi^{\tau_E(\rho)(z)})^*L$  for  $z \in \partial D^Y$ , and  $u(\zeta_0^Y) = (\phi^{\tau(\rho)})^*\gamma$ , where  $\tau$  is the conformal factor that  $\mathbf{K}^{D,Y}$  assigns to  $\zeta_0^Y$ .
- (3)  $u(p_D(\rho)) \in Y_\sigma(\rho)$ .

Gromov compactness applies to  $\mathcal{D}^Y(\gamma)$ , and in fact the situation is better than expected. Namely, suppose  $\rho_i$  is a sequence with  $p_D(\rho_i) < 1$  but  $\lim p_D(\rho_i) = 1$ , and that  $u_i \in \mathcal{D}_{\rho_i}^Y(\gamma)$  is a Gromov convergent sequence. Then we expect the Gromov limit to contain a bubble component with the incidence condition. However, in this case everything is exact, so all bubbles are constant. Thus, the incidence condition on the bubble is equivalent to an incidence condition on  $\partial D^Y$ , which means the incidence condition is a point of  $L \cap Y_\sigma(\lim \rho_i)$ . However,  $Y_\sigma(\rho)$  lies in the image of  $\sigma$  whenever  $p_D(\rho) = 1$ , while  $L$  is an interior Lagrangian, which means that no such point exists. This shows that, in fact,  $\mathcal{D}_\rho^Y(\gamma)$  is empty for  $p_D(\rho)$  sufficiently close to 1.

Applying the usual transversality argument, we now obtain

**Lemma 5.18.** *There is a comeager subset  $\mathcal{K}_{\text{reg}}^{D,Y}(M[\sigma]) \subset \mathcal{K}^{D,Y}(M[\sigma])$  such that, for every family  $\mathbf{K}^{D,Y} \in \mathcal{K}_{\text{reg}}^{D,Y}(M[\sigma])$ , the following hold.*

- (1) For all interior Lagrangians  $L$  and all  $\gamma \in \mathcal{X}(L, L)$ ,  $\mathcal{D}^Y(\gamma)$  is a smooth manifold of dimension  $\deg(\gamma)$ . It is empty if  $n_\sigma(\gamma) > 0$ .
- (2) If  $\deg(\gamma) = 0$ , then  $\mathcal{D}^Y(\gamma)$  is compact.
- (3) If  $\deg(\gamma) = 1$ , then  $\mathcal{D}^Y(\gamma)$  has a Gromov compactification  $\overline{\mathcal{D}}^Y(\gamma)$  which is a compact topological 1-manifold with boundary, and there is a canonical identification

$$\partial \overline{\mathcal{D}}^Y(\gamma) = \coprod_{\tilde{x} \in \mathcal{X}(H_t)} \mathcal{R}_1^{0+1}(\tilde{x}; \gamma) \times \mathcal{C}(\tilde{x}) \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L, L)} \mathcal{R}(\tilde{\gamma}; \gamma) \times \mathcal{D}^Y(\tilde{\gamma})$$

□

Fix  $\mathbf{K}^{D,Y} \in \mathcal{K}_{\text{reg}}^{D,Y}(M[\sigma])$ . We now repeat the above with one input. Choose a diffeomorphism  $\Psi: [0, 1] \rightarrow \overline{\mathcal{R}}_1^{1+1}$  such that  $\Psi(0)$  is nodal at the first positive puncture of  $\Sigma^{2+1} \in \mathcal{R}^{2+1}$  and  $\Psi(1)$  is nodal at the second positive puncture of  $\Sigma^{2+1}$ . Let  $Z^Y$  be the strip  $\mathbb{R} \times [0, 1]$ , where  $\mathbb{R} \times \{i\}$  is  $\partial_i Z^Y$ , and the ends  $-\infty$  and  $+\infty$  are labeled  $\zeta_0^Y$  and  $\zeta_1^Y$ , respectively. Then  $\Psi$  induces a  $(0, 1)$ -parametrized family of strip-like ends  $\epsilon_q^{Z,Y}$  on  $Z^Y$  by specifying, for  $q \in (0, 1)$ , the embedding  $\Psi(q) \hookrightarrow Z^Y$  which sends  $\partial \Psi(q)$  to  $\partial Z^Y$ ,  $\zeta_i$  to  $\zeta_i^Y$ , and  $\zeta_+$  to a point on  $\{0\} \times (0, 1)$ . Extend this to a  $(0, 1] \times (0, 1)$ -parametrized family  $\epsilon^{Z,Y}$  with the following properties

- (1) For  $(\rho, q) \in (0, 1] \times (0, 1)$  with  $\rho$  small or  $q \leq \frac{1}{4}$  or  $q \geq \frac{3}{4}$ ,  $\epsilon^{Z,Y}(\rho, q)$  agrees with  $\epsilon_q^{Z,Y}$ .
- (2) For  $\rho$  close to 1 and  $q \in [\frac{1}{4}, \frac{3}{4}]$ ,  $\epsilon^{Z,Y}(\rho, q)$  agrees up to shift with the canonical strip-like ends on  $Z$ .

Similarly, choose a smooth  $(0, \frac{1}{2}] \times (0, 1)$ -parametrized family of finite cylinders  $\delta^{Z,Y}$  on  $Z^Y$  as follows.

- (1) For  $(\rho, q) \in (0, 1] \times (0, 1)$  with  $\rho$  small,  $\delta^{Z,Y}(\rho, q)$  agrees with the finite cylinder obtained by gluing  $\epsilon^f$  on  $\mathbb{C}$  to  $\epsilon_+$  on  $\Psi(q)$  with length  $e^{\frac{1}{\rho}}$ .

- (2) For  $q$  close to 0 or 1,  $\delta^{Z,Y}(\rho, q)$  agrees with the finite cylinder induced by gluing  $D^Y(\rho)$  to the appropriate input of  $\Sigma^{2+1}$  with length dictated by consistency with  $\Psi$ .
- (3) For  $\rho = \frac{1}{2}$  and any  $q$ , the cylinder  $\delta^{Z,Y}(\rho, q)$  has width zero.

We then think of this data as a  $(0, 1] \times (0, 1)$ -parametrized space of Riemann surfaces with boundary, ends, and cylinders  $Z^Y(\rho, q)$ , which for  $\rho \in (0, \frac{1}{2}]$  is equipped with the strip-like ends  $\epsilon^{Z,Y}$  and finite cylinder  $\delta^{D,Y}(\rho)$  and for  $\rho > \frac{1}{2}$  is equipped only with  $\epsilon^{Z,Y}$ .

Consider the space  $\mathcal{K}^{Z,Y}(M[\sigma])$  of smooth,  $(0, 1] \times (0, 1)$ -parametrized families of Floer data  $\mathbf{K}^{Z,Y}$  on  $Z^Y$  with the following properties.

- (1) For  $(\rho, q) \in (0, 1] \times (0, 1)$  with  $\rho$  small,  $\mathbf{K}^{Z,Y}(\rho, q)$  extends smoothly to  $\rho = 0$ , where it agrees up to conformal equivalence with the Floer data chosen for  $\mathbb{C}$  and  $\Psi(q)$ .
- (2) For  $q$  close to 0 or 1,  $\mathbf{K}^{Z,Y}(\rho, q)$  is conformally close in the sense of Definition 3.10 to the Floer datum induced by gluing  $D^Y(\rho)$  to the appropriate input of  $\Sigma^{2+1}$  with length dictated by consistency with  $\Psi$ .
- (3) For  $\rho = 1$  and  $q \in [\frac{1}{4}, \frac{3}{4}]$ ,  $\mathbf{K}^{Z,Y}(\rho, q)$  is conformally equivalent to the Floer perturbation  $(H^{0,1}, dt, 1)$  which gives rise to the Floer differential.

Finally, we choose a function  $p_Z: (0, 1] \times (0, 1) \rightarrow Z^Y$  extending  $p_D$  in the following sense. For  $q$  near 0 or 1,  $p_D(\rho)$  can be thought of as a point on the thick part of  $D^Y$ , and we require  $p_Z(\rho, q)$  to be the point of  $Z^Y$  which corresponds to  $p_D(\rho)$  under gluing. For  $\rho$  close to 0,  $p_Z(\rho, q)$  agrees with the point of  $Z^Y(\rho, q)$  coming from the origin in  $\mathbb{C}$  under the gluing of  $\mathbb{C}$  to  $\Psi(q)$ . For  $\rho$  close to 1, we require that  $p_Z(\rho, q)$  is  $\rho$ -independent and depends on  $q$  in the following way. For  $q \leq \frac{1}{3}$  or  $q \geq \frac{2}{3}$ ,  $p_Z(\rho, q)$  is on the boundary of  $Z^Y$ . For  $q \in [\frac{1}{3}, \frac{2}{3}]$ , the  $[0, 1]$ -component of  $p_Z$  increases monotonically from 0 to 1.

Let  $L_0$  and  $L_1$  be interior Lagrangians of  $M[\sigma]$ , and let  $\gamma_0, \gamma_1 \in \mathcal{X}(L_0, L_1)$ . For a universal choice  $\mathbf{K}^{Z,Y} \in \mathcal{K}^{Z,Y}(M[\sigma])$ , the corresponding space of holomorphic strips is called  $\mathcal{Z}^Y(\gamma_1; \gamma_0)$  and is the union over all  $(\rho, q) \in (0, 1] \times (0, 1)$  of the spaces  $\mathcal{Z}_{\rho,q}^Y(\gamma)$  of maps

$$u: Z^Y \rightarrow \widehat{M[\sigma]}$$

satisfying the conditions

- (1)  $u$  satisfies (4.12) for  $\mathbf{K}^{Z,Y}(\rho, q)$ .
- (2)  $u(z) \in (\phi^{\tau_E(\rho)}(z))^* L_i$  for  $z \in \partial_i D^Y$ , and  $u(\zeta_i^Y) = (\phi^{\tau_i(\rho, q)})^* \gamma_i$ , where  $\tau_i$  is the conformal factor that  $\mathbf{K}^{Z,Y}$  assigns to  $\zeta_i^Y$ .
- (3)  $u(p_Z(\rho, q)) \in Y_\sigma(\rho)$ .

The compactness situation is the same as before, and we have

**Lemma 5.19.** *There is a comeager subset  $\mathcal{K}_{reg}^{Z,Y}(M[\sigma]) \subset \mathcal{K}^{Z,Y}(M[\sigma])$  such that, for every universal choice  $\mathbf{K}^{Z,Y} \in \mathcal{K}_{reg}^{Z,Y}(M[\sigma])$ , the following hold.*

- (1) For all interior Lagrangians  $L_0, L_1$  and all  $\gamma_0, \gamma_1 \in \mathcal{X}(L_0, L_1)$ ,  $\mathcal{Z}^Y(\gamma_1; \gamma_0)$  is a smooth manifold of dimension  $\deg(\gamma_0) - \deg(\gamma_1) + 1$ . It is empty if  $n_\sigma(\gamma_0) > n_\sigma(\gamma_1)$ .
- (2) If  $\deg(\gamma_0) - \deg(\gamma_1) = -1$ , then  $\mathcal{Z}^Y(\gamma_1; \gamma_0)$  is compact.
- (3) If  $\deg(\gamma_0) - \deg(\gamma_1) = 0$ , then  $\mathcal{Z}^Y(\gamma_1; \gamma_0)$  has a Gromov compactification  $\overline{\mathcal{Z}}^Y(\gamma_1; \gamma_0)$  which is a compact topological 1-manifold with boundary, and its boundary is in natural bijection

with

$$\begin{aligned}
& \coprod_{\tilde{x} \in \mathcal{X}(H_t)} \mathcal{R}_1^{1+1}(\tilde{x}, \gamma_1; \gamma_0) \times \mathcal{C}(\tilde{x}) \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_0, L_1)} \mathcal{Z}^Y(\tilde{\gamma}; \gamma_0) \times \mathcal{R}(\gamma_1; \tilde{\gamma}) \\
& \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_1, L_1)} \mathcal{R}^{2+1}(\tilde{\gamma}, \gamma_1; \gamma_0) \times \mathcal{D}^Y(\tilde{\gamma}) \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_0, L_0)} \mathcal{R}^{2+1}(\gamma_1, \tilde{\gamma}; \gamma_0) \times \mathcal{D}^Y(\tilde{\gamma}) \\
& \amalg \coprod_{\tilde{\gamma} \in \mathcal{X}(L_0, L_1)} \mathcal{R}(\tilde{\gamma}; \gamma_0) \times \mathcal{Z}^Y(\gamma_1; \tilde{\gamma}) \amalg \coprod_{\substack{t \in (0,1) \\ \gamma_0(t) \in Y_\sigma(1)}} \tilde{\mathcal{R}}(\gamma_1; \gamma_0).
\end{aligned}$$

Of course, by Lemma 3.3, the last term only occurs when  $\gamma_0 = \gamma_1$ .  $\square$

As with the other homotopies, there is a filtered version of  $\mathcal{Z}^Y(\gamma_1; \gamma_0)$ . This is obtained by choosing, for all  $(\rho, q)$ , a path  $e: [0, 1] \rightarrow Z^Y$  such that  $e(0) \in \partial_0 Z^Y$  and  $e(1) = p_Z(\rho, q)$ . The filtered component  $\mathcal{Z}_{filt}^Y(\gamma_1; \gamma_0)$  consists of all  $u \in \mathcal{Z}^Y(\gamma_1; \gamma_0)$  such that  $u$  avoids  $D_\sigma$  and  $u \circ e$  has topological intersection number zero with  $\sigma(\hat{F} \times \mathbb{R}_+)$ . The condition that  $Y_\sigma(\rho)$  avoids  $\sigma(\hat{F} \times \mathbb{R}_{\geq 0})$  ensures that this is indeed a connected component of  $\mathcal{Z}^Y(\gamma_1; \gamma_0)$ .

Fixing  $\mathbf{K}^{Z,Y} \in \mathcal{K}_{reg}^{Z,Y}(M[\sigma])$ , define a linear map  $h_{f_\sigma}^3: \text{hom}_{\sigma'}^*(L_0, L_1) \rightarrow \text{hom}_{\sigma'}^{*-1}(L_0, L_1)$  by

$$\begin{aligned}
(5.42) \quad h_{f_\sigma}^3(\gamma^m \otimes \cdots \otimes \gamma^0) &= \sum_{\substack{0 \leq i \leq m+1 \\ n_\sigma(\gamma^r) = 0 \forall r < i \\ \tilde{\gamma} \text{ making the result composable} \\ \deg(\tilde{\gamma}) = 0}} \# \mathcal{D}^Y(\tilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^i \otimes \tilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0 \\
&+ \sum_{\substack{0 \leq i \leq m \\ n_\sigma(\gamma^r) = 0 \forall r < i \\ \tilde{\gamma} \text{ making the result composable} \\ \deg(\tilde{\gamma}) = \deg(\gamma^i) - 1}} \# \mathcal{Z}_{filt}^Y(\gamma^i; \tilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^{i+1} \otimes \tilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0.
\end{aligned}$$

As always (cf. Remark 5.6), one needs to exclude certain bad terms to make  $h_{f_\sigma}^3$  well defined. The argument for this is essentially the same as before, with the moving incidence condition handled as in Remark 4.27.

We are finally rewarded for the bizarre filtered moduli spaces:

**Lemma 5.20.** *Let  $f_\sigma$  be a saddle unit, and let  $\gamma = \gamma^m \otimes \cdots \otimes \gamma^0 \in \text{hom}_{\sigma'}^*(L_0, L_1)$  have  $\sum_{i=0}^m n_\sigma(\gamma_i) > 0$ . Then, up to terms which decrease the main filtration,*

$$(5.43) \quad (h_{f_\sigma}^3 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 h_{f_\sigma}^3)(\gamma) = \mathcal{CO}_{f_\sigma}^{filt}(\gamma) + \gamma$$

*Proof.* Following the usual argument, we obtain

$$(h_{f_\sigma}^3 \mu_{\sigma'}^1 + \mu_{\sigma'}^1 h_{f_\sigma}^3)(\gamma) = \mathcal{CO}_{f_\sigma}^{filt}(\gamma) + \sum_{\substack{0 \leq i \leq m \\ n_\sigma(\gamma^r) = 0 \forall r < i}} \sum_{\substack{t \in (0,1) \\ \gamma^i(t) \in Y_\sigma(1) \\ \gamma^i(t') \notin \sigma(\hat{F} \times \mathbb{R}_+) \text{ for any } t' < t}} \# \tilde{\mathcal{R}}(\gamma^i; \gamma^i) \cdot \gamma.$$

The coefficient  $\# \tilde{\mathcal{R}}(\gamma^i; \gamma^i)$  is of course 1, but we include it for clarity. Examining the conditions on the sums, we see that the only  $i$  which contributes is the smallest  $i$  such that  $n_\sigma(\gamma^i) \neq 0$ . For this  $\gamma^i$ , let  $t_0 \in (0, 1)$  be the first time at which  $\gamma^i$  intersects  $\sigma(\hat{F} \times \mathbb{R}_+)$ . Since  $\gamma^i$  starts outside the image of  $\sigma$ , it crosses  $Y_\sigma(1)$  topologically once before  $t_0$ , and hence the sum contributes a total coefficient of 1.  $\square$

For  $y \in CC_*(\mathcal{B}_\sigma(\sigma))$  closed of low action with  $\mathcal{OC}(y) = f_\sigma$  a saddle unit, set

$$h_y := h_y^1 + h_y^2 + h_{f_\sigma}^3.$$

This is the last ingredient we need to prove the stop removal formula:

*Proof of Theorem 4.10.* Since  $\sigma$  is strongly nondegenerate, Corollary 4.28 provides us for all sufficiently large  $\mathbf{m}$  with a cycle  $y_{\mathbf{m}} \in CC_*(\mathcal{B}_{\sigma}(\sigma))$  of small action such that  $\mathcal{OC}(y_{\mathbf{m}})$  is a saddle unit for Floer datum  $K^{\mathbb{C}, \mathbf{m}}$ . Define a linear map  $\Delta_{y_{\mathbf{m}}}: \text{hom}_{\sigma'}^*(L_0, L_1) \rightarrow \text{hom}_{\sigma'}^{*-1}(L_0, L_1)$  by

$$\Delta_{y_{\mathbf{m}}}(\gamma) = \begin{cases} \Delta_{y_{\mathbf{m}}}^0 + h_{y_{\mathbf{m}}} & \text{if } (\mathbf{n}, \mathbf{m})(\gamma) > (1, 0) \\ 0 & \text{otherwise.} \end{cases}$$

As noted in Remarks 5.6 and 5.11, this is well defined for sufficiently large  $\mathbf{m}$ . Write  $y = y_{\mathbf{m}}$  for such an  $\mathbf{m}$ , and let  $R_y = \text{id} + \mu_{\sigma'}^1 \Delta_y + \Delta_y \mu_{\sigma'}^1$  be the basic retraction as in Section 5.1. It suffices to show that  $R_y$  satisfies the conditions of Lemma 5.3, as this would prove Proposition 5.1 and hence Theorem 4.10. Condition (1) is trivial. To prove Condition (2), combine Equation (5.15) with Lemmas 5.15, 5.17, and 5.20.  $\square$

## APPENDIX A. ENERGY AND COMPACTNESS

To prove the various compactness claims, we need to know that, given a set of input chords or orbits, there are only finitely many possible outputs, and that these constrain the resulting holomorphic curves to some compact subset of our ambient space  $M$ . Appendix A.1 addresses the first part of this. For the second, we use Appendix A.2 away from cylindrical ends and interpolate to Ganatra’s setup ([19] Appendix A) near those. Our maximum principle coincides with his for the “unperturbed region” on a separating open set, and so they patch together. All of the included proofs are essentially standard.

**A.1. Action inequalities.** Let  $\Sigma$  be a Riemann surface with boundary components  $\partial_i \Sigma$ , interior punctures  $z_j$ , and boundary punctures  $\zeta^k$ . Let  $\epsilon_j$  be cylindrical ends at  $z_j$ , and likewise let  $\epsilon^k$  be strip-like ends at  $\zeta^k$ , such that the images of the ends are pairwise disjoint.

Suppose further we are given a Liouville domain  $(M, \lambda_M)$ , and that  $\partial_i \Sigma$  is labeled with a smooth transverse family of Lagrangians  $\{L_i(z) \in \hat{M} \mid z \in \partial_i \Sigma\}$  which is constant in the strip-like ends. Here, the term “transverse” is in the sense of Section 3.3. Additionally, we have a  $\Sigma$ -parametrized compatible Hamiltonian  $\tilde{H}$ , a  $\Sigma$ -parametrized perturbing Hamiltonian  $P$ , along with 1-forms  $\beta$  and  $\beta^\ell$  with the following properties.

- (1) On the cylindrical and strip-like ends, let  $t$  be the coordinate of the compact  $S^1$  or  $[0, 1]$  factor, and let  $s$  be the coordinate on the  $\mathbb{R}_{\geq 0}$  or  $\mathbb{R}_{\leq 0}$  factor. Then  $\tilde{H}$  is both  $s$ - and  $t$ -independent,  $P$  is  $s$ -independent, and  $\beta = wdt$  for some positive constant  $w$  depending on the end.
- (2)  $d\beta \leq 0$  and  $\beta|_{\partial \Sigma} = 0$ .
- (3) Outside of a compact set,  $d^\Sigma \tilde{H} \wedge \beta \leq 0$ , and further there is some  $c < 0$  such that  $\tilde{H}d\beta + d^\Sigma \tilde{H} \wedge \beta \leq c$  on the support of  $\beta^\ell$ .
- (4) For any  $z \in \partial_i \Sigma$  and  $\xi \in T_z \partial \Sigma$ , the vector field  $\beta^\ell(\xi) \cdot X_{\sqrt{\tilde{H}(z)}}$  is tangent outside of a compact set to the deformation of  $L_i(z)$  associated to  $\xi$ .
- (5)  $P$  is globally bounded, and  $\|X_P\|$  decays exponentially in the symplectization coordinate  $\sqrt{\tilde{H}}$  for any metric coming from some  $J \in \mathcal{J}^\Sigma(M, \tilde{H})$ .
- (6)  $d^\Sigma P \wedge \beta \leq 0$ , and the supports of  $\beta^\ell$  and  $dP$  are disjoint.

Finally, we suppose we have some  $\Sigma$ -parametrized  $\hat{\omega}_M$ -compatible almost complex structure  $J$ .

*Remark A.1.* In the setup of the main text, the 1-form  $\beta^\ell$  was only used for relating isotopic Lagrangians (section 3.3), while the perturbing Hamiltonian  $P$  was only used in the cylindrical ends and the finite cylinders (section 5). This means that condition 6 above is trivially satisfied.



The other conditions correspond individually to conditions we imposed on Floer data throughout the text, see in particular Definitions 3.8, 3.18, and 4.15.

Set  $H = \tilde{H} + P$ . We consider the space  $\mathcal{M}$  of maps  $u: \Sigma \rightarrow \hat{M}$  sending every  $z \in \partial_i \Sigma$  to a point  $L_i(z)$  and satisfying the inhomogeneous Cauchy-Riemann equation

$$J \circ (du - X_H \otimes \beta - X_{\sqrt{\tilde{H}}} \otimes \beta^\ell) = (du - X_H \otimes \beta - X_{\sqrt{\tilde{H}}} \otimes \beta^\ell) \circ j$$

where  $j$  is the almost-complex structure on  $\Sigma$ . Given a Kähler metric on  $\Sigma$ , define the **geometric energy** of such  $u$  as

$$E^{geom}(u) = \int_{\Sigma} \|du - X_H \otimes \beta - X_{\sqrt{\tilde{H}}} \otimes \beta^\ell\|^2 dvol,$$

where the metric on  $\hat{M}$  is  $\Sigma$ -dependent and is obtained from  $J$ . This is independent of the choice of Kähler metric on  $\Sigma$  and is in fact given by

$$E^{geom}(u) = \int_{\Sigma} \left( u^* \hat{\omega}_M - u^*(d^{\hat{M}} H) \wedge \beta - u^*(d^{\hat{M}} \sqrt{\tilde{H}}) \wedge \beta^\ell \right).$$

In fact, this formula would hold and the theory would go through if  $\sqrt{\tilde{H}}$  were replaced with any other linear Hamiltonian  $H^\ell$  satisfying  $[H, H^\ell] = 0$ , but it is hard to find such  $H^\ell$ .

We also define the **topological energy** of  $u$  by

$$\begin{aligned} E^{top}(u) &= \int_{\Sigma} \left( u^* \hat{\omega}_M - d(u^* H \beta) - d(u^* \sqrt{\tilde{H}} \beta^\ell) \right) \\ &= E^{geom}(u) - \int_{\Sigma} \left( u^* H d\beta + u^*(d^{\Sigma} H) \wedge \beta + u^* \sqrt{\tilde{H}} d\beta^\ell + u^*(d^{\Sigma} \sqrt{\tilde{H}}) \wedge \beta^\ell \right). \end{aligned}$$

**Lemma A.2.** *There is some constant  $C \in \mathbb{R}$  depending only on  $\Sigma$ ,  $\tilde{H}$ ,  $P$ ,  $\beta$ , and  $\beta^\ell$  for which*

$$E^{top}(u) \geq E^{geom}(u) + C$$

for all  $u \in \mathcal{M}$ .

*Proof.* We begin by noting that  $Hd\beta$ ,  $d^{\Sigma} H \wedge \beta$ ,  $\sqrt{\tilde{H}} d\beta^\ell$ , and  $d^{\Sigma} \sqrt{\tilde{H}} \wedge \beta^\ell$  all belong to  $\Omega^2(\Sigma; C^\infty(\hat{M}))$  and are compactly supported in the  $\Sigma$ -direction. Moreover,  $Hd\beta$  and  $d^{\Sigma} \tilde{H} \wedge \beta$  are asymptotically quadratic in the symplectization coordinate  $r$ , while  $d^{\Sigma} \sqrt{\tilde{H}} \wedge \beta^\ell$  and  $\sqrt{\tilde{H}} d\beta^\ell$  are linear in  $r$  and  $d^{\Sigma} P \wedge \beta$  is nonpositive by assumption. Thus, condition 3 above shows that

$$Hd\beta + (d^{\Sigma} H) \wedge \beta + \sqrt{\tilde{H}} d\beta^\ell + (d^{\Sigma} \sqrt{\tilde{H}}) \wedge \beta^\ell$$

is nonpositive outside of a compact subset of  $\Sigma \times \hat{M}$ . This means that it is bounded above, and because it is compactly supported in the  $\Sigma$ -direction this gives the desired conclusion.  $\square$

We now examine the relation between action and topological energy. For any  $u \in \mathcal{M}$  with  $E^{geom}(u) < \infty$ ,  $u$  converges at  $\epsilon_j$  to a 1-periodic orbit  $x_j$  of  $w_j X_H$  and at  $\epsilon^k$  to a chord  $\gamma^k$  of  $w^k X_H$ , for appropriate 1-parameter specializations of  $H$ . Before we define the action, choose for each boundary component  $\partial_i \Sigma$  a smooth family of functions  $f_i(z) \in C^\infty(L_i(z))$  which satisfy  $d[f_i(z)] = \hat{\lambda}_M|_{L_i(z)}$  and are independent of  $z$  in the strip-like ends. For an orbit  $x$ , define

$$A(x) = \int_{S^1} \left( x^* \hat{\lambda}_M - w x^* H dt \right).$$

For a chord  $\gamma$ , let  $L_0$  and  $L_1$  be the Lagrangians containing  $\gamma(0)$  and  $\gamma(1)$ , respectively, and let  $f_0$  and  $f_1$  be the primitives chosen above for  $L_0$  and  $L_1$ . Define

$$\begin{aligned} A_0(\gamma) &= \int_{[0,1]} \left( \gamma^* \hat{\lambda}_M - w \gamma^* H dt \right) \\ A(\gamma) &= \int_{[0,1]} \left( \gamma^* \hat{\lambda}_M - w \gamma^* H dt \right) + f_0(\gamma(0)) - f_1(\gamma(1)). \end{aligned}$$

With this set up, we have

**Lemma A.3.** *There is some constant  $D \in \mathbb{R}$  depending only  $\Sigma$ ,  $\tilde{H}$ ,  $\beta^\ell$ ,  $\{L_i\}$ , and  $\{f_i\}$  for which*

$$E^{top}(u) \leq \sum_{\substack{\text{positive} \\ \text{cylindrical} \\ \text{ends } \epsilon_j}} A(x_j) + \sum_{\substack{\text{positive} \\ \text{strip-like} \\ \text{ends } \epsilon^k}} A(\gamma^k) - \sum_{\substack{\text{negative} \\ \text{cylindrical} \\ \text{ends } \epsilon_j}} A(x_j) - \sum_{\substack{\text{negative} \\ \text{strip-like} \\ \text{ends } \epsilon^k}} A(\gamma^k) + D$$

for all  $u \in \mathcal{M}$ .

*Proof.* To begin, pick an element  $\eta \in \Omega^1(\partial\Sigma; \Gamma(T\hat{M}))$  which agrees with  $X_{\sqrt{\tilde{H}}} \beta^\ell$  outside of a compact subset of  $\partial\Sigma \times \hat{M}$  and such that, for all  $z \in \partial_i \Sigma$  and  $\xi \in T_z \partial\Sigma$ , the vector field  $\eta(z)(\xi)$  is tangent to the deformation of  $L_i(z)$  associated to  $\xi$ . For any  $u \in \mathcal{M}$  of finite energy,  $du|_{\partial\Sigma} - \eta(u(z))$  is then valued at  $z \in \partial_i \Sigma$  in vector fields tangent to the Lagrangian submanifold  $L_i(z)$ .

Next, for each  $i$ , let  $L_{(i)}$  be the smooth manifold underlying  $L_i(z)$ , so that we may view  $L_i(z)$  as a family of exact Lagrangian embeddings  $\Lambda_i(z): L_{(i)} \hookrightarrow \hat{M}$ . Moreover, we may arrange that for any vector field  $\xi \in \Gamma(T\partial_i \Sigma)$ , we have

$$\mathcal{L}_\xi \Lambda_i = \eta(\xi)$$

After pulling back by  $\Lambda_i$ , we may view  $f_i(z)$  as functions on  $L_{(i)} \times \partial_i \Sigma$ .

For any  $u \in \mathcal{M}$  of finite energy, set

$$\begin{aligned} A_0(u) &= \sum_{\substack{\text{positive} \\ \text{cylindrical} \\ \text{ends } \epsilon_j}} A(x_j) + \sum_{\substack{\text{positive} \\ \text{strip-like} \\ \text{ends } \epsilon^k}} A_0(\gamma^k) - \sum_{\substack{\text{negative} \\ \text{cylindrical} \\ \text{ends } \epsilon_j}} A(x_j) - \sum_{\substack{\text{negative} \\ \text{strip-like} \\ \text{ends } \epsilon^k}} A_0(\gamma^k) \\ A(u) &= \sum_{\substack{\text{positive} \\ \text{cylindrical} \\ \text{ends } \epsilon_j}} A(x_j) + \sum_{\substack{\text{positive} \\ \text{strip-like} \\ \text{ends } \epsilon^k}} A(\gamma^k) - \sum_{\substack{\text{negative} \\ \text{cylindrical} \\ \text{ends } \epsilon_j}} A(x_j) - \sum_{\substack{\text{negative} \\ \text{strip-like} \\ \text{ends } \epsilon^k}} A(\gamma^k). \end{aligned}$$

We evaluate

$$\begin{aligned}
E^{top}(u) &= \int_{\partial\bar{\Sigma}} \left( u^* \hat{\lambda}_M - u^* H \beta - u^* \sqrt{\widetilde{H}} \beta^\ell \right) \\
&= A_0(u) + \int_{\partial\Sigma} \left( u^* \hat{\lambda}_M - u^* \sqrt{\widetilde{H}} \beta^\ell \right) \\
&= A_0(u) + \int_{\partial\Sigma} \left( \langle \hat{\lambda}_M, du - \eta(u) \rangle + \langle \hat{\lambda}_M, \eta(u) \rangle - u^* \sqrt{\widetilde{H}} \beta^\ell \right) \\
&\leq A_0(u) + \sum_i \int_{\partial_i \Sigma} \left( \langle d^{L(i)} f_i, du - \eta(u) \rangle + \langle \hat{\lambda}_M, X_{\sqrt{\widetilde{H}}}(u) \beta^\ell \rangle - u^* \sqrt{\widetilde{H}} \beta^\ell \right) + D_1 \\
&\leq A_0(u) + \sum_i \int_{\partial_i \Sigma} \langle d^{L(i)} f_i, du - \eta(u) \rangle + D_2 \\
&= A_0(u) + \sum_i \int_{\partial_i \Sigma} u^* d^{L(i)} f_i + D_2 \\
&= A_0(u) + \sum_i \int_{\partial_i \Sigma} \left( u^* df_i - d^{\partial_i \Sigma} f_i(u) \right) + D_2 \\
&= A(u) - \sum_i \int_{\partial_i \Sigma} d^{\partial_i \Sigma} f_i(u) + D_2
\end{aligned}$$

As a 1-form on  $\partial_i \Sigma$ , this final integrand is compactly supported and globally bounded independently of  $u$ , so it can be bounded by an additive constant. This completes the proof.  $\square$

Because  $E^{geom}(u) \geq 0$  for any  $u \in \mathcal{M}$ , Lemmas A.2 and A.3 give an upper bound on the action of the output chords and orbits in terms of the data on  $\Sigma$  and the action of the input chords and orbits. To obtain finiteness, we note that for any orbit  $x$  on the portion of  $\hat{M}$  where  $H$  is quadratic, we have

$$\begin{aligned}
A(x) &= \int_{S^1} \left( x^* \hat{\lambda}_M - wx^* H dt \right) \\
&= \int_{S^1} \left( \hat{\lambda}_M(w X_H) dt - wx^* H dt \right) \\
&= w \int_{S^1} \left( x^* (2\widetilde{H}) dt - x^* \widetilde{H} dt \right) + w \int_{S^1} \left( \hat{\lambda}_M(X_P) dt - x^* P dt \right) \\
&= w \int_{S^1} \widetilde{H}(x) dt + w \int_{S^1} \left( \hat{\lambda}_M(X_P) - x^* P \right) dt.
\end{aligned}$$

The first integral is very nearly the same as  $w\widetilde{H}(x(t_0))$ , while the second is globally bounded. In particular  $A$  is proper and bounded below on the space of  $X_H$ -orbits. A similar result holds for  $A(\gamma)$  or  $A_0(\gamma)$ . Thus, the space of possible outputs for a given choice of inputs is compact. Since our Hamiltonians are nondegenerate, this shows that it is finite.

Aside from proving finiteness, Section 4 relies on stronger action bounds with controllable leaks. To accomplish this, we use that  $\beta^\ell$  vanishes in this setting and that  $L_i(z) = (\phi^{\tau_E(z)})^* L_i^0$ , where  $\tau_E: \partial\Sigma \rightarrow \mathbb{R}_{>0}$  is a positive rescaling function and  $L_i^0$  is a fixed Lagrangian associated to  $\partial_i \Sigma$ . Here, as usual when discussing rescalings, we use  $\phi^\tau$  to denote the time  $\log \tau$  Liouville flow rather than the time  $\tau$  Liouville flow.

To begin, note that

**Lemma A.4.** *If  $K$  is a Floer datum on  $\Sigma$ , then the action of a chord or orbit with respect to  $K$  is  $\tau$  times the action of the time  $\log \tau$  Liouville pullback of that chord or orbit with respect to the Floer datum  $K_\tau$ .*

*Remark on proof.* It is a short computation to check that the integrals defining the action have the right scaling properties. For strips, one needs to add in the term associated to the primitives on  $L_0$  and  $L_1$ . There, one uses that if  $f_i$  is a primitive for  $\lambda_M|_{L_i}$ , then  $\frac{f_i}{\tau}$  is a primitive for  $\lambda_M|_{(\phi\tau)^*L_i}$ .  $\square$

Following this train of thought, we may universally choose  $\frac{f_i}{\tau_E(z)}$  as the primitive for  $L_i(z)$ , where  $f_i$  is a fixed primitive for  $L_i^0$ . This allows us to recast Lemma A.3 as an equality.

**Lemma A.5.** *Every  $u \in \mathcal{M}$  with  $E^{\text{geom}}(u) < \infty$  satisfies*

$$E^{\text{top}}(u) = \sum_{\substack{\text{positive} \\ \text{cylindrical} \\ \text{ends } \epsilon_j}} A(x_j) + \sum_{\substack{\text{positive} \\ \text{strip-like} \\ \text{ends } \epsilon^k}} A(\gamma^k) - \sum_{\substack{\text{negative} \\ \text{cylindrical} \\ \text{ends } \epsilon_j}} A(x_j) - \sum_{\substack{\text{negative} \\ \text{strip-like} \\ \text{ends } \epsilon^k}} A(\gamma^k) + \int_{\partial\Sigma} \frac{(\phi^{\tau_E} \circ u)^* f_L}{\tau_E^2} d\tau_E.$$

*Proof.* Stokes's theorem.  $\square$

Combining this with Lemma A.2 and positivity of geometric energy, we get a bound for output action.

**Corollary A.6.** *Suppose  $\Sigma$  has exactly one negative end  $\epsilon_-$ , either strip-like or cylindrical. Let  $u \in \mathcal{M}$  be a finite-energy holomorphic curve converging at  $\epsilon_-$  to an orbit or chord  $y$ . Then*

$$A(y) \leq \sum_{\substack{\text{positive} \\ \text{cylindrical} \\ \text{ends } \epsilon_j}} A(x_j) + \sum_{\substack{\text{positive} \\ \text{strip-like} \\ \text{ends } \epsilon^k}} A(\gamma^k) + \int_{\Sigma} \max_{\hat{M}}(d^{\Sigma}H \wedge \beta) + \|f_L\|_{C^0} \int_{\partial\Sigma} \frac{|d\tau_E|}{\tau_E^2}.$$

$\square$

**A.2. A maximum principle.** With data as above, we further assume that  $P = 0$ ,  $J$  is asymptotically  $\hat{Z}_M$ -invariant on the contact planes  $\ker dH \cap \ker \hat{\lambda}_M$ , and that there is some compact subset  $K \subset \hat{M}$  such that, outside  $\Sigma \times K$ ,

$$(A.1) \quad d^{\hat{M}}H \circ J = -g(H)\hat{\lambda}_M$$

for some strictly positive function  $g$ . Additionally, we require that there is some  $k$  such that  $d\beta < k < 0$  on the supports of  $\partial^{\Sigma}H$  and  $\beta^{\ell}$ . In the main text, these conditions are all achieved outside the 2-shifts of the cylinders, and that is where we wish to prove a maximum principle. In the 2-shifts,  $J$  is of rescaled contact type, and hence Ganatra's estimates provide a  $C^0$  bound on any holomorphic curve  $u: \Sigma \rightarrow \hat{M}$  depending only on the Floer data and the actions of the asymptotic ends. Once we have a maximum principle outside the 2-shifts of the cylinders, we will conclude that any maximum of  $u$  outside some fixed compact subset of  $\hat{M}$  must occur in the closure of the 2-shifts of the cylinders, and so it will be bounded by a constant depending, again, on the Floer data and the actions of the asymptotic ends.

Note that  $H(z)$  is quadratic for all  $z \in \Sigma$ , which means we can write  $d^{\Sigma}H = H\nu$  for some  $\nu \in \Omega^1(\Sigma, C^{\infty}(\hat{M}))$ . Moreover, we may enlarge  $K$  to assume  $\nu$  is  $\hat{Z}_M$ -invariant on the complement of  $\Sigma \times K$ .

We are interested in two regimes. In the first,  $\partial^{\Sigma}H = 0$  and  $\beta^{\ell} = 0$ , but  $g$  is allowed to be  $\Sigma$ -dependent. In the second,  $g(x) = cx$  for some  $c > 0$ , and no assumptions are made on  $\partial^{\Sigma}H$  or on  $\beta^{\ell}$ . Note that in the main text, condition 4b of Definition 3.8 guarantees that  $\Sigma$  can be divided into regions falling into one of the two regimes. In either case, we will use the Hopf maximum principle, which states that if some function  $F: \Sigma \rightarrow \mathbb{R}$  satisfies  $\Delta F \geq 0$  modulo  $dF$ , then the local maxima of  $F$  all occur on  $\partial\Sigma$ , and that at such local maxima the outward normal derivative of  $F$

is strictly positive. We take  $F = H \circ u$ , and compute

$$\begin{aligned}
d^c F &= d^\Sigma H(u) \circ j + d^{\hat{M}} H \circ du \circ j \\
&= F\nu(u) \circ j + d^{\hat{M}} H \circ (du - X_H \beta - X_{\sqrt{H}} \beta^\ell) \circ j \\
&= F\nu(u) \circ j - g(F) \hat{\lambda}_M \circ (du - X_H \beta - X_{\sqrt{H}} \beta^\ell) \\
&= F\nu(u) \circ j - g(F) \cdot \left( u^* \hat{\lambda}_M - 2F\beta - \sqrt{F} \beta^\ell \right).
\end{aligned}$$

In the first regime,  $\nu = 0$  and  $\beta^\ell = 0$ , so that modulo  $dF$  we have

$$\begin{aligned}
\Delta F \cdot dvol &= -dd^c F \\
&= d^\Sigma g(F) \wedge \left( u^* \hat{\lambda}_M - 2F\beta \right) + g(F) \cdot (u^* \hat{\omega}_M - 2Fd\beta) \\
&= d^\Sigma g(F) \wedge \frac{-1}{g(F)} d^c F + g(F) \cdot \left( [u^* \hat{\omega}_M - u^*(d^{\hat{M}} H) \wedge \beta - u^*(d^{\hat{M}} \sqrt{H}) \wedge \beta^\ell] \right. \\
&\quad \left. + u^*(d^{\hat{M}} H) \wedge \beta - 2Fd\beta \right).
\end{aligned}$$

The  $d^c F$  term is linear in  $dF$  and can be ignored, and in the first regime  $dH = d^{\hat{M}} H$ , so modulo  $dF$  we are left with

$$\begin{aligned}
\Delta F \cdot dvol &= g(F) \cdot \left( \|du - X_H \beta - X_{\sqrt{H}} \beta^\ell\|^2 + u^*(dH) \wedge \beta - 2Fd\beta \right) \\
&= g(F) \cdot \left( \|du - X_H \beta - X_{\sqrt{H}} \beta^\ell\|^2 + dF \wedge \beta - 2Fd\beta \right) \\
&= g(F) \cdot \left( \|du - X_H \beta - X_{\sqrt{H}} \beta^\ell\|^2 - 2Fd\beta \right).
\end{aligned}$$

The right hand side is globally nonnegative, so the Hopf maximum principle applies.

In the second regime, we instead have

$$d^c F = F\nu \circ j - cF u^* \hat{\lambda}_M + 2cF^2 \beta + cF^{\frac{3}{2}} \beta^\ell.$$

For convenience of notation, we will write in local coordinates

$$-\nu(u) \circ j = c\nu_1(s, t, u(s, t))ds + c\nu_2(s, t, u(s, t))dt,$$

where  $\nu_1$  and  $\nu_2$  are functions on  $\Sigma \times \hat{M}$  which are  $\hat{Z}_M$ -invariant outside  $\Sigma \times K$ . Modulo  $dF$ , this gives

$$\begin{aligned}
-dd^c F &= cF \cdot \left( (D_1\nu_2 - D_2\nu_1)ds \wedge dt + \langle D_3\nu_1, du \rangle \wedge ds + \langle D_3\nu_2, du \rangle \wedge dt \right. \\
&\quad \left. + u^*\hat{\omega}_M - 2Fd\beta - \sqrt{F}d\beta^\ell \right) \\
&= cF \cdot \left( \|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\|^2 + u^*(d^{\hat{M}}H) \wedge \beta + u^*(d^{\hat{M}}\sqrt{H}) \wedge \beta^\ell \right. \\
&\quad \left. + \langle D_3\nu_1, du \rangle \wedge ds + \langle D_3\nu_2, du \rangle \wedge dt - 2Fd\beta - \sqrt{F}d\beta^\ell + (D_1\nu_2 - D_2\nu_1)ds \wedge dt \right) \\
&= cF \cdot \left( \|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\|^2 + dF \wedge \beta + d\sqrt{F} \wedge \beta^\ell - d^\Sigma H(u) \wedge \beta - d^\Sigma \sqrt{H}(u) \wedge \beta^\ell \right. \\
&\quad + \langle D_3\nu_1, du - X_H\beta - X_{\sqrt{H}}\beta^\ell \rangle \wedge ds + \langle D_3\nu_2, du - X_H\beta - X_{\sqrt{H}}\beta^\ell \rangle \wedge dt \\
&\quad + \langle D_3\nu_1, X_H\beta + X_{\sqrt{H}}\beta^\ell \rangle \wedge ds + \langle D_3\nu_2, X_H\beta + X_{\sqrt{H}}\beta^\ell \rangle \wedge dt \\
&\quad \left. - 2Fd\beta - \sqrt{F}d\beta^\ell + (D_1\nu_2 - D_2\nu_1)ds \wedge dt \right) \\
&= cF \cdot \left( \|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\|^2 \right. \\
&\quad + \langle D_3\nu_1, du - X_H\beta - X_{\sqrt{H}}\beta^\ell \rangle \wedge ds + \langle D_3\nu_2, du - X_H\beta - X_{\sqrt{H}}\beta^\ell \rangle \wedge dt \\
&\quad + \langle D_3\nu_1, X_H\beta + X_{\sqrt{H}}\beta^\ell \rangle \wedge ds + \langle D_3\nu_2, X_H\beta + X_{\sqrt{H}}\beta^\ell \rangle \wedge dt \\
&\quad \left. - 2Fd\beta - F\nu(u) \wedge \beta - \frac{1}{2}\sqrt{F}\nu(u) \wedge \beta^\ell - \sqrt{F}d\beta^\ell + (D_1\nu_2 - D_2\nu_1)ds \wedge dt \right).
\end{aligned}$$

Now  $D_3\nu_1$  and  $D_3\nu_2$  vanish on  $\hat{Z}_M$  and are  $\hat{Z}_M$ -invariant. Since the metric  $\hat{\omega}_M(\cdot, J\cdot)$  grows with  $H$  in the  $\partial M$  directions,  $\|D_3\nu_1\|$  and  $\|D_3\nu_2\|$  tend to zero as  $H$  tends to infinity. Thus, we can apply Cauchy-Schwarz and obtain that, modulo  $dF$ ,

$$\begin{aligned}
-dd^c F &\geq cF \cdot \left( \|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\|^2 dvol - \frac{1}{c}\|D_3\nu\| \|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\| dvol \right. \\
&\quad + \langle D_3\nu_1, X_H\beta + X_{\sqrt{H}}\beta^\ell \rangle \wedge ds + \langle D_3\nu_2, X_H\beta + X_{\sqrt{H}}\beta^\ell \rangle \wedge dt \\
&\quad \left. - 2Fd\beta - F\nu(u) \wedge \beta - \frac{1}{2}\sqrt{F}\nu(u) \wedge \beta^\ell - \sqrt{F}d\beta^\ell + (D_1\nu_2 - D_2\nu_1)ds \wedge dt \right) \\
&= cF \cdot \left( (\|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\| - \frac{1}{2c}\|D_3\nu\|)^2 dvol - 2Fd\beta - F\nu(u) \wedge \beta \right. \\
&\quad + \langle D_3\nu_1, X_H\beta + X_{\sqrt{H}}\beta^\ell \rangle \wedge ds + \langle D_3\nu_2, X_H\beta + X_{\sqrt{H}}\beta^\ell \rangle \wedge dt \\
&\quad \left. - \frac{1}{2}\sqrt{F}\nu(u) \wedge \beta^\ell - \sqrt{F}d\beta^\ell + (D_1\nu_2 - D_2\nu_1)ds \wedge dt - \frac{1}{4c^2}\|D_3\nu\|^2 dvol \right)
\end{aligned}$$

On the other hand,  $-2Fd\beta$  and  $-F\nu(u) \wedge \beta$  are assumed to be nonnegative, and moreover  $-2Fd\beta$  grows faster than any term in the last two lines and is assumed to be strictly positive on the support of those terms. Thus, the right hand side is nonnegative for sufficiently large  $F$ . In particular, we obtain

**Lemma A.7.** *In the above setup, suppose there is some  $k$  such that  $d\beta < k < 0$  on the supports of  $d^\Sigma H$  and  $\beta^\ell$ , and that*

$$\text{support}(d^\Sigma g) \cap \left( \text{support}(d^\Sigma H) \cup \text{support}(\beta^\ell) \right) = \emptyset.$$

*Then there is some  $R$  depending on  $g$ ,  $H$ ,  $\beta$ , and  $\beta^\ell$ , but not on  $u$ , for which  $u^*H$  satisfies the Hopf maximum principle outside of  $H^{-1}((-\infty, R))$ .  $\square$*

It remains to prevent maxima on  $\partial\Sigma$ . For this, let  $\xi$  be a vector field along  $\partial\Sigma$  which points in the negative direction, so that  $j\xi$  points outward. We calculate

$$\begin{aligned} dF(j\xi) &= d^c F(\xi) = d^\Sigma H(u)(j\xi) - g(F) \cdot \left( \hat{\lambda}_M(du(\xi)) - 2F\beta(\xi) - \sqrt{F}\beta^\ell(\xi) \right) \\ &= d^\Sigma H(u)(j\xi) - g(F)\hat{\lambda}_M \left( du(\xi) - X_{\sqrt{H}}\beta^\ell(\xi) \right) \\ &= d^\Sigma H(u)(j\xi). \end{aligned}$$

This gives

**Lemma A.8.** *In the situation of Lemma A.7, if additionally  $d^\Sigma H$  vanishes on outward normal vectors at  $\partial\Sigma$ , then  $u^*H$  has no local maxima outside  $H^{-1}((-\infty, R))$ . In other words,  $u^*H$  is bounded by the larger of  $R$  and the values of  $H$  on the asymptotic  $X_H$  chords and orbits.*

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